Transfer Matrix Theory for a Type of Uniaxial Layers: Starting from Basic Electromagnetism

Robert J. Steed

13/02/13

Abstract

A review of the relatively standard electromagnetism concepts needed for a transfer matrix calculation (modelling the reflection and refraction of a stack of thin dielectric layers). This covers, the wave equation, complex refractive indices and complex dielectric constants (for modelling absorption), the different forms of the Fresnel Equations and the derivation of the transfer matrix formulism. There is also a part on the modelling of uniaxial materials, concentrating on uniaxial layers with their optical axis perpendicular to the interfaces. A final part covers polaritons and quantum well intersubband transitions, including the depolarisation shift effect.

Contents

Introduction

These notes were developped in parallel with the code in the pyLuminous library (python modules for modelling dielectric layers). They were initially for the benefit of my lousy memory and therefore they are quite terse. Hence they are mostly for useful for those who want to see the theory behind the transfer matrix calculation rather than a way to learn optics/electromagnetism theory. I hope that someone finds this useful.

Part I summarises basic electromagnetism and Part II covers the transfer matrix calculations. Part III details the theory of uniaxial layers with their optical axis perpendicular to the interface, this theory is harder to find in the optics books. It also describes how quantum well optical transitions can be modelled. Finally Part IV introduces strongly coupled transitions and their desciption in terms of polaritons. At the end there is a set of references.

Part I

Classical Electromagnetism

I will try to stick to SI notations in the following sections (i.e. no c.g.s.). I'm not going to reference everything since most of this will very standard.

1 Maxwell's Equations in Media

These are Maxwell's equations in a medium.

$$\begin{aligned} \underline{\nabla}.\mathbf{D} &= \rho_f && (1)\\ \underline{\nabla}.\mathbf{B} &= 0 && (2)\\ \underline{\nabla}\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} && (3)\\ \underline{\nabla}\times\mathbf{H} &= \frac{\partial\mathbf{D}}{\partial t}+\mathbf{J}_f && (4) \end{aligned}$$

and the continuity equation for charge and current is

$$\underline{\nabla}.\mathbf{J}_f+\frac{\partial\rho_f}{\partial t}=0 \tag{5}$$

Bound and free charges and currents are split up so that the response of medium can be treated using constitutive relations between D and E and H and B. So we have

$$\begin{aligned} \rho_b &= \underline{\nabla}.\mathbf{P} && (6)\\ \rho &= \rho_f+\rho_b && (7) \end{aligned}$$

$$\begin{aligned} \mathbf{J}_b &= \underline{\nabla}\times\mathbf{M}+\frac{\partial\mathbf{P}}{\partial t} && (8)\\ \mathbf{J} &= \mathbf{J}_f+\mathbf{J}_b && (9) \end{aligned}$$

In general, **D** and **H** are given by

$$\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P} \tag{10}$$

$$\mu_0\left(\mathbf{H}+\mathbf{M}\right)=\mathbf{B} \tag{11}$$

however, often it is found that the polarization of the medium can be described by

$$\mathbf{P}=\varepsilon_0\chi\mathbf{E} \tag{12}$$

where χ is the material's susceptibility (which is most often a function of frequency) where

$$\varepsilon_r = 1 + \chi$$

Normally for optics, $\mathbf{M} = 0$ and so the constitutive relations are

$$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E} \tag{13}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} \tag{14}$$

where ε_r can be a function of time and space, a function of the fields themselves in the case of non-linear materials or a second rank tensor for anisotropic materials. If ε_r is a function of frequency (normally the case), it becomes a convolution as a function of time

$$\begin{aligned} \mathbf{D} &= \varepsilon_0 \varepsilon_r * \mathbf{E} \qquad (15) \\ &= \varepsilon_0 \int_{-\infty}^{\infty} \varepsilon_r(\tau)\, \mathbf{E}(t-\tau)\, d\tau \end{aligned}$$

If the materials are anisotropic, the constitutive relations become

$$\mathbf{D} = \varepsilon_0 \underline{\varepsilon_r} \mathbf{E} \tag{16}$$

$$\mu_0 \underline{\mu_r} \mathbf{H} = \mathbf{B} \tag{17}$$

where $\underline{\varepsilon_r}$ and $\underline{\mu_r}$ are matrices. This formulisation also allows the description of gyrotropic media for which orthogonal circular polarisations travel at different speeds (and in some cases this even breaks reciprocity).

It is possible for the constitutive relations to become even more complicated. For example ε_r may depend upon the value of $\mathbf{E}$ from other locations or times leading to hysteresis and it is even possible for ε_r to depend upon both $\mathbf{E}$ and $\mathbf{H}$. The magnetic permeability μ_r may possess equivalent properties.

Lastly, connecting microscopic models of a system to its macroscopic properties is usually more complicated than it first appears. This is the topic of effective media (see sec.6).

1.1 Wave Equation

A triple vector product of div can be simplified

$$\underline{\nabla} \times (\underline{\nabla} \times \mathbf{f}) = \nabla(\nabla.\mathbf{f}) - \nabla^2 \mathbf{f} \tag{18}$$

therefore $\nabla\times$eqn.3 leads to

$$\underline{\nabla}(\underline{\nabla}.\mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial(\underline{\nabla} \times \mathbf{B})}{\partial t} \tag{19}$$

Then using eqn.1 but assuming that there are no frees charges, we can eliminate the first term. Using eqn.4 with eqn.14 and assuming that there are no free currents, the right hand side of the equation becomes a second time derivative of $\mathbf{D}$

$$\nabla^2 \mathbf{E} = \mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} \tag{20}$$

Assuming that $\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$ where the dielectric constant is constant, we can take ε_r outside of the time derivative

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{21}$$

Likewise, $\nabla\times$eqn.4 leads to

$$\nabla^2 \mathbf{B} = \mu_0 \frac{\partial(\underline{\nabla} \times \mathbf{D})}{\partial t} \tag{22}$$

In an isotropic medium $\nabla \times (\varepsilon_r \mathbf{E}) = (\nabla \varepsilon_r) \times \mathbf{E} + \varepsilon_r \nabla \times \mathbf{E}$ and if the medium is also homogeneous then $\nabla \times (\varepsilon_r \mathbf{E}) = \varepsilon_r \nabla \times \mathbf{E}$ then

$$\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial(\mathbf{B})}{\partial t} \tag{23}$$

These electric and magnetic wave equations are tightly linked using eqn.3 or eqn.4, hence the term electromagnetic wave.

1.2 Wave Equation: derivation 2 - Dispersive Media

Considering $\nabla\times$eqn.3 leads to

$$\underline{\nabla}\left(\underline{\nabla}.\mathbf{E}\right)-\nabla^2\mathbf{E}=-\frac{\partial\left(\underline{\nabla}\times\mathbf{B}\right)}{\partial t} \tag{24}$$

Then using eqn.1 and assuming that there are no frees charges, we have

$$\varepsilon_0\varepsilon_r * \underline{\nabla}.\mathbf{E} \;=\; 0 \tag{25}$$

and so we can eliminate the first term from eqn.24. This implies that $\mathbf{k}.\mathbf{E}=0$ for waves in a lossy medium. Using eqn.4 with eqn.14 and assuming that there are no free currents, the right hand side of the equation becomes a second time derivative of $\mathbf{D}$

$$\nabla^2\mathbf{E}=\mu_0\frac{\partial^2\mathbf{D}}{\partial t^2} \tag{26}$$

Assuming that $\mathbf{D}=\varepsilon_0\varepsilon_r * \mathbf{E}$ where the dielectric constant is a complex valued function with frequency dependence, we can still as usual take ε_r outside of the time derivative

$$\nabla^2\mathbf{E}=\mu_0\varepsilon_0\varepsilon_r * \frac{\partial^2\mathbf{E}}{\partial t^2} \tag{27}$$

Likewise, $\nabla\times$eqn.4 and following through as before leads to

$$\nabla^2\mathbf{B}=\mu_0\frac{\partial\left(\underline{\nabla}\times\mathbf{D}\right)}{\partial t} \tag{28}$$

In an isotropic medium $\nabla\times(\varepsilon_r\mathbf{E})=(\nabla\varepsilon_r)\times\mathbf{E}+\varepsilon_r\nabla\times\mathbf{E}$ and if the medium is also homogeneous then $\nabla\times(\varepsilon_r\mathbf{E})=\varepsilon_r\nabla\times\mathbf{E}$ then

$$\nabla^2\mathbf{B}=\mu_0\varepsilon_0\varepsilon_r * \frac{\partial\mathbf{B}}{\partial t} \tag{29}$$

These electric and magnetic wave equations are tightly linked using eqn.3 or eqn.4, hence the term electromagnetic wave.

2 Dielectric Constant and Refractive Index

From the wave equation, eqn.20, we look for solutions of the form[1], $\exp\left(i\left(\mathbf{k}.\mathbf{r}-\omega t\right)\right)$. Remembering that this is complex notation, the real electric field is the real part of this term. For a homogeneous dielectric medium this gives

$$\mathbf{k}^2=\mu_0\varepsilon_0\varepsilon_r\omega^2$$

at this point c is defined

$$c=\frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

and refractive index

$$n=\sqrt{\varepsilon_r}$$

The dispersion relation for light is then

$$\frac{\omega}{|\underline{\mathbf{k}}|}=\frac{c}{n} \tag{30}$$

2.1 Lossy Media, complex epsilons and complex refractive indices

Consider a medium that possesses conductivity. Ohm's law gives

$$\mathbf{J}=\sigma\mathbf{E} \tag{31}$$

If we rederive the wave equation without assuming that $\mathbf{J}=0$, we now get

$$\nabla^2\mathbf{E}=\mu_0\varepsilon_0\underline{\varepsilon_r} * \frac{\partial^2\mathbf{E}}{\partial t^2}+\mu_0\sigma\frac{\partial\mathbf{E}}{\partial t} \tag{32}$$

[1] or if you are an engineer $\exp\left(j\left(\omega t-\mathbf{k}.\mathbf{r}\right)\right)$

This gives a dispersion relation of

$$\mathbf{k}^2 = \mu_0\varepsilon_0\varepsilon_r\omega^2 + i\mu_0\sigma\omega \tag{33}$$

which can be recast as a complex dielectric permittivity as[2]

$$\varepsilon_r = \varepsilon_r + i\frac{\sigma}{\omega\varepsilon_0} \tag{34}$$

Or we can define a complex dieletric permittivity aka. dielectric constant

$$\varepsilon_r = \varepsilon' + i\varepsilon'' \tag{35}$$

At optical frequencies, I find it unhelpful to think of absorption features as high frequency conductivity and so derivations will use the above equation as a starting point. In fact, in the Lorentz oscillator model a complex permittivity naturally emerges without reference to **J**, see sec.6.1. In these cases, the complex term emerges from the time derivative of the polarisation of the medium in the definition of **D** see eqn.10.

Similarly we can define a complex refractive index that includes absorption[3][4]

$$n = n' + i\kappa \tag{36}$$

We can relate the two definitions, leading to

$$\begin{aligned} \varepsilon' &= n'^2 - \kappa^2 && (37) \\ \varepsilon'' &= 2n'\kappa && (38) \end{aligned}$$

and

$$\begin{aligned} n' &= \sqrt{\frac{\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}}{2}} && (39) \\ \kappa &= \sqrt{\frac{-\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}}{2}} && (40) \end{aligned}$$

By defining things in this way, we implicitly assume that we are using a complex representation of the electromagnetic waves. In these situations, k will generally be complex. If we want to use real quantities, then we have to go back to the original dispersion relations.

Now we have

$$\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \Rightarrow \underline{\mathbf{E}}_0 \exp\left(-\frac{\kappa}{c}\omega\hat{\mathbf{k}}\cdot\mathbf{r}\right)\exp\left(i\frac{n'}{c}\omega\hat{\mathbf{k}}\cdot\mathbf{r} - i\omega t\right) \tag{41}$$

where $\hat{\mathbf{k}}$ is the k vector with unit length. The intensity is related to electric field by $I \propto |\mathbf{E}|^2$ and so we can compare the above equation with Beer's law (eqn.379) and we can see that

$$\alpha = \frac{2\omega\kappa}{c} \tag{42}$$

Attenuation is also described sometimes by a penetration depth or skin depth. Sometimes by other terms too! The wikipedia page Mathematical_descriptions_of_opacity covers the multitude of definitions quite thoroughly.

2.2 Kramers Kronig Relations

We can connect the real and imaginary parts of $\varepsilon$and n using the Kramers Kronig relations which are a consequence of the condition that causality holds true.

[2] or $\varepsilon_r = \varepsilon_r + j\frac{\sigma}{\omega}$ if you're an engineer.

[3] engineering definitions change the plus to a minus sign.

[4] But Born & Wolf define it as $n = n'(1 + i\kappa)$ so everything changes.

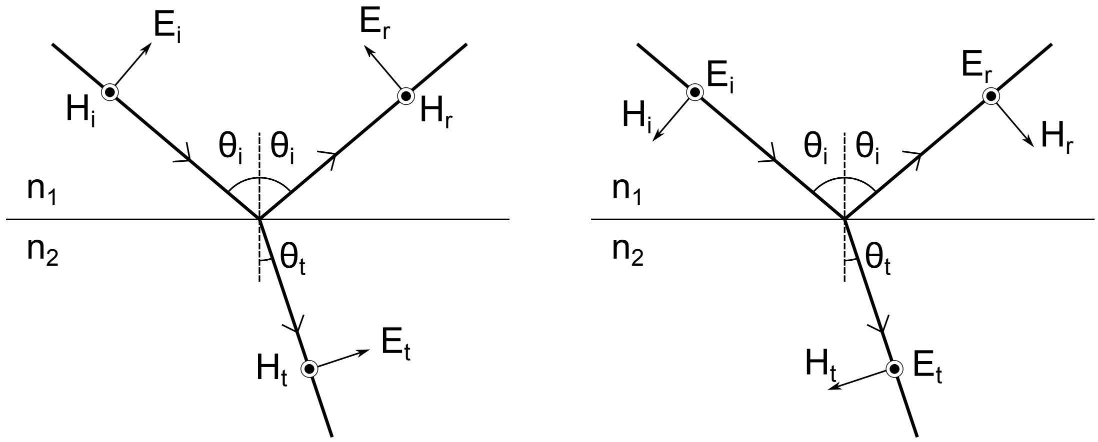


Figure 1: Definitions of field directions for the Fresnel coefficients presented here. The left diagram is for the electric field in the plane of incidence, p-polarisation or TM polarisation. The right diagram is for the electric field perpendicular to the plane of incidence, s-polarisation or TE polarisation.

3 Poynting Vectors and Irradiances

The Poynting vector describes the flow of radiative energy, it appears in the Poynting theorem which is a statement of the conservation of energy

$$\frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{S} - \mathbf{J}_f \cdot \mathbf{E} \tag{43}$$

where $\mathbf{J}_f$ is the current density of free charges, u is the electromagnetic energy density

$$u = \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}\right) \tag{44}$$

and $\mathbf{S}$ is the Poynting vector, normally defined as

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{45}$$

although there has been a long running controversy over the definition of the vector within dielectric media and there exists a different form involving D and B (the Minkowski form). Strictly speaking, there are limits to the validity of the Poynting theorem within an arbitray medium which can cause problems. In addition, the Poynting theorem is unchanged by the addition of a curl of a field to the Poynting vector and so the Poynting vector is not strictly defined which has been another source of controversy.

For formulisms using complex vector quantities, it can be shown that the time averaged Poynting vector is given by

$$\mathbf{S} = \frac{1}{2}\Re\left[\mathbf{E} \times \mathbf{H}^*\right]$$

4 Fresnel Equations

The Fresnel equations give the amplitudes of the transmitted and reflected wave for a plane wave incident on a dielectric interface. All the optics books derive these equations[1, 2, 3, 4, 5]. The derivations are interesting and bring up points about surface charges and currents. We also need to take care with sign definitions, since there is a choice for the definition of the direction of the reflected wave i.e. some versions have $r_p \to -r_p$ compared to the equations here.

4.1 Snell's Law

Snell's law relates the directions of travel for the optical waves in the media either side of the interface. Hence, if the angle of incidence is θ_i then angle of the travel of the light leaving the interface (θ_t) is given by

$$n_1 \sin\theta_i = n_2 \sin\theta_t \tag{46}$$

where n_1 and n_2 are the refractive indices of the two media. Snell's law can be shown in various ways and also drops naturally out of most derivations of the Fresnel equations

4.2 Set a

s-polarised (TE mode) reflection coefficient[2]

$$r_s = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} \tag{47}$$

p-polarised (TM mode) reflection coefficient

$$r_p = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_1 \cos\theta_t + n_2 \cos\theta_i} \tag{48}$$

It is this last equation in particular that depends on the conventions chosen when the initial equations are defined. In many books one will find this equation is negated.

The reflection coefficient is the ratio of the reflected to the incident electric field amplitudes. The reflectivity or reflectance[5] relates to the reflected intensity

$$R = |r|^2 \tag{49}$$

The transmission coefficients are

$$t_s = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t} \tag{50}$$

$$t_p = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_t + n_2 \cos\theta_i} \tag{51}$$

The transmittance or transmissivity has to take into account the change in beam cross section due to the change in beam angle in the material and change in speed of the wave, hence

$$T = \left(\frac{n_2 \cos\theta_t}{n_1 \cos\theta_i}\right) |t|^2 \tag{52}$$

Alternatively, one can use snell's law on the original equations to get

$$r_s = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \tag{53}$$

$$r_p = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \tag{54}$$

$$t_s = \frac{2 \sin\theta_t \cos\theta_i}{\sin(\theta_t + \theta_i)} \tag{55}$$

$$t_p = \frac{4 \sin\theta_t \cos\theta_i}{\sin(2\theta_t) - \sin(2\theta_i)} \tag{56}$$

4.3 Set b

Often the Fresnel equations are given in simpler forms. For instance, putting

$$m = \frac{\cos\theta_t}{\cos\theta_i} \quad \text{and} \quad \rho = \frac{n_2}{n_1} \tag{57}$$

leads to

$$r_s = \frac{1 - \rho m}{1 + \rho m} \tag{58}$$

$$r_p = \frac{\rho - m}{m + \rho} \tag{59}$$

$$t_s = \frac{2}{1 + m\rho} \tag{60}$$

$$t_p = \frac{2}{m + \rho} \tag{61}$$

[5] I'm pretty sure that these terms are strictly defined somewhere.

4.4 Set c

Alternative simplications seen are [5]

$$p = \sqrt{\frac{\varepsilon_r}{\mu_r}}\cos\theta \quad \text{for s-pol or TE mode} \tag{62}$$

$$q = \sqrt{\frac{\mu_r}{\varepsilon_r}}\cos\theta \quad \text{for p-pol or TM mode} \tag{63}$$

leading to

$$r_s = \frac{p_1 - p_2}{p_1 + p_2} \tag{64}$$

$$r_p = \frac{q_1 - q_2}{q_1 + q_2} \tag{65}$$

$$t_s = \frac{2p_1}{p_1 + p_2} \tag{66}$$

$$t_p = \frac{2\frac{n_1}{n_2}q_1}{q_1 + q_2} \tag{67}$$

defining $\Lambda = \frac{p_2}{p_1}$ or $\Lambda = \frac{q_2}{q_1}$, we finally have

$$r = \frac{1-\Lambda}{1+\Lambda} \tag{68}$$

$$t_s = \frac{2}{1+\Lambda} \tag{69}$$

$$t_p = \frac{2}{1+\Lambda}\frac{n_1}{n_2} \tag{70}$$

The last equation seems stranges but actually makes the final form for the transmission simpler

$$T = \frac{4\Lambda}{(1+\Lambda)^2} \tag{71}$$

for both polarisations. Some people redefine t_p and it's relation to T to make the two polarisations symmetric. In this case

$$r = \frac{1-\Lambda}{1+\Lambda} \tag{72}$$

$$t = \frac{2}{1+\Lambda} \tag{73}$$

$$T = \Lambda\,|t|^2 \tag{74}$$

for both polarisations. However, this does make the p-pol (TM) transmission coefficient wrong by the lost factor

4.5 Set d

Staelin [3] names things slightly differently in terms of admittances or impediances. I think that Staelin defines the medium constants in absolute terms. ie. substitute $\mu = \mu_r\mu_0$ and $\varepsilon = \varepsilon_r\varepsilon_0$ to compare to my other equations.

For TE waves, he gives

$$\Gamma_{TE} = \frac{Z_n^{TE} - 1}{Z_n^{TE} + 1} \tag{75}$$

$$T_{TE} = \frac{2Z_n^{TE}}{Z_n^{TE} + 1} \tag{76}$$

$$Z_n^{TE} = \frac{\mu_t/k_{tz}}{\mu_i/k_{iz}} \tag{77}$$

where Z_n^{TE} is the impediance of the medium. At normal incidence this reduces to

$$Z_n^{TE} = \frac{\eta_t}{\eta_i} \tag{78}$$

where η is the impedance of the medium

$$\eta_i = \sqrt{\frac{\mu_i}{\varepsilon_i}} \tag{79}$$

and often

$$\frac{E}{H} = \eta \tag{80}$$

Then for TM waves, he gives

$$\Gamma_{TM} = -\frac{Y_n^{TM} - 1}{Y_n^{TM} + 1} \tag{81}$$

$$T_{TM} = \frac{2}{Y_n^{TM} + 1} \tag{82}$$

$$Y_n^{TM} = \frac{\varepsilon_t / k_{tz}}{\varepsilon_i / k_{iz}} = \frac{1}{Z_n^{TM}} \tag{83}$$

where Y_n^{TM} is the admittance of the medium.

4.6 Set e

We can even write Fresnel's equations using k-vectors. Using k_1 and k_2 for the 1st and 2nd media respectively.

$$r_s = \frac{k_{1\perp} - k_{2\perp}}{k_{1\perp} + k_{2\perp}} \tag{84}$$

$$r_p = \frac{\varepsilon_2 k_{1\perp} - \varepsilon_1 k_{2\perp}}{\varepsilon_2 k_{1\perp} + \varepsilon_1 k_{2\perp}} \tag{85}$$

$$t_s = \frac{2k_{1\perp}}{k_{1\perp} + k_{2\perp}} \tag{86}$$

$$t_p = \frac{2\varepsilon_2 k_{1\perp}}{\varepsilon_2 k_{1\perp} + \varepsilon_1 k_{2\perp}} \frac{n_1}{n_2} \tag{87}$$

where

$$k_{1_{\shortparallel}} = k_{2_{\shortparallel}} = k_{\shortparallel} \tag{88}$$

$$k_{1\perp}^2 = k_1^2 - k_{1_{\shortparallel}}^2 \tag{89}$$

$$= \left(\frac{\omega n_1}{c}\right)^2 - k_{\shortparallel}^2 \tag{90}$$

As usual

$$T = \left(\frac{k_{2\perp}}{k_{1\perp}}\right) |t|^2 \tag{91}$$

If we define $\Lambda = \frac{k_{2\perp}}{k_{1\perp}}$ for s-pol and $\Lambda = \frac{\varepsilon_1 k_{2\perp}}{\varepsilon_2 k_{1\perp}}$ for p-pol, then the relations become (as before).

$$r_s = \frac{1 - \Lambda}{1 + \Lambda} \tag{92}$$

$$r_p = \frac{1 - \Lambda}{1 + \Lambda} \tag{93}$$

$$t_s = \frac{2}{1 + \Lambda} \tag{94}$$

$$t_p = \frac{2}{1 + \Lambda} \frac{n_1}{n_2} \tag{95}$$

and $T = \frac{4\Lambda}{(1+\Lambda)^2}$ for both polarisations.

5 Waves in Lossy Media

In fact it is usually very easy to consider lossy medium, all that we need to do is describe the lossy medium with a complex dielectric constant (aka. permitivity) or equivalently using a complex refractive. This often happens quite naturally when modelling the dielectric constant (see sec.6.1). The relationships between these and other formulisms is covered in sec.2. Then looking at Snell's law,

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

it seems that we can still find solutions as long as we allow θ_t to be a complex number. Remarkably, it then turns out that we can continue to use the Fresnel equations unchanged as long as we use this complex angle! It still seems remarkably fortuitious to me that this works but it makes everything quite straight forward.

5.1 Inhomogeneous Waves

If you ever want to model the waves within an absorbing layer, you will need to realise that they are inhomogeneous, that means that the direction in which waves decay in amplitude is not the same as the direction of the wavefronts. This can be inferred from symmetry considerations if we consider an infinite parallel wave leaving an interface at an angle. The wave can have a direction at an angle to the interface but it must decay perpendicular to the interface by symmetry.

Taking the wave equation eqn.20 and looking for plane wave solutions, we obtain

$$\left(\underline{\mathbf{k}}' + i\underline{\mathbf{k}}''\right)^2 = (n + i\kappa)^2 \frac{\omega^2}{c^2} \tag{96}$$

or equivalently

$$\left(\underline{\mathbf{k}}' + i\underline{\mathbf{k}}''\right)^2 = \left(\varepsilon' + i\varepsilon''\right) \frac{\omega^2}{c^2} \tag{97}$$

Now, often this is simplified to

$$k' + ik'' = (n + i\kappa)\frac{\omega}{c} \tag{98}$$

but this assumes that the two vectors are colinear. More generally

$$k'^2 + 2i\underline{\mathbf{k}}'' \cdot \underline{\mathbf{k}}' - k''^2 = \left(\varepsilon' + i\varepsilon''\right) \frac{\omega^2}{c^2} \tag{99}$$

leading to

$$k'^2 + 2ik''k' \cos\varphi - k''^2 = \left(\varepsilon' + i\varepsilon''\right) \frac{\omega^2}{c^2} \tag{100}$$

or

$$k'^2 + 2ik''k' \cos\varphi - k''^2 = \left(n'^2 - \kappa^2 + 2in'\kappa\right) \frac{\omega^2}{c^2} \tag{101}$$

Giving solutions

$$\kappa^2 = \frac{c^2}{2\omega^2}\left[\left(k'^2 - k''^2\right) \pm \sqrt{\left(k'^2 - k''^2\right)^2 + \left(2k'k'' \cos\varphi\right)^2}\right] \tag{102}$$

$$n'^2 = \frac{c^2}{2\omega^2}\left[-\left(k'^2 - k''^2\right) \pm \sqrt{\left(k'^2 - k''^2\right)^2 + \left(2k'k'' \cos\varphi\right)^2}\right] \tag{103}$$

or

$$k'^2 = \frac{\omega^2}{2c^2}\left[\left(n'^2 - \kappa^2\right) \pm \sqrt{\left(n'^2 - \kappa^2\right)^2 + \left(\frac{2n'\kappa}{\cos\varphi}\right)^2}\right] \tag{104}$$

$$k''^2 = \frac{\omega^2}{2c^2}\left[-\left(n'^2 - \kappa^2\right) \pm \sqrt{\left(n'^2 - \kappa^2\right)^2 + \left(\frac{2n'\kappa}{\cos\varphi}\right)^2}\right] \tag{105}$$

These solutions are known as inhomogeneous waves[3]. They occur when a plane wave enters a lossy medium. But what is the angle φ?

5.2 Transmittance for a Lossy Media

We previously defined the Transmittance for a lossy medium as eqn.52

$$T = \left(\frac{n_2 \cos\theta_t}{n_1 \cos\theta_i}\right) |t|^2$$

or alternatively by eqn.71

$$T = \frac{4\Lambda}{|1+\Lambda|^2}$$

but we see that for absorbing layers, this will lead to a complex Transmittance which hardly seems physical. This isn't very important usually but when we want to model a set of optical thick layers using an incoherent version of the transfer matrix theory that works with intensities rather than fields, this problem becomes an issue.

I have seen various redefinitions that are meant to fix this problem but the issue is often confusing, so I've seen

$$T = \frac{\Re\left[n_2 \cos\theta_t\right]}{\Re\left[n_1 \cos\theta_i\right]} |t|^2$$

and I've seen

$$T = \Re\left(\frac{n_2 \cos\theta_t}{n_1 \cos\theta_i}\right) |t|^2$$

but these are only equivalent if n_1 and θ_i are real since in general $\Re\left[\frac{z_2}{z_1}\right] \neq \frac{\Re[z_2]}{\Re[z_1]}$.

One possible approach is to insist that energy is conserved at the boundary

$$R + T = 1$$

This can fufilled if we redefine the transmission coefficient as [6]

$$T_s = \Re\left(\frac{n_2 \cos\theta_t}{n_1 \cos\theta_i}\right) |t_s|^2 \tag{106}$$

for s-polarisation (TE) while p-polarisation (TM) is now defined as

$$T_p = \Re\left(\frac{n_2^* \cos\theta_t}{n_1^* \cos\theta_i}\right) |t_p|^2 \tag{107}$$

This can equivalently be written for both polarisations as

$$T = \Re\left[\Lambda\right] \left|\frac{2}{1+\Lambda}\right|^2 \tag{108}$$

and we can easily show that

$$R + T = \left|\frac{1-\Lambda}{1+\Lambda}\right|^2 + \Re\left[\Lambda\right] \left|\frac{2}{1+\Lambda}\right|^2 = 1$$

This derivation also works when we consider a beam incident from the other side of the interface since $\Lambda_b = \frac{1}{\Lambda}$

$$\begin{aligned} T_b &= \Re\left[\frac{1}{\Lambda}\right] \left|\frac{2\Lambda}{\Lambda+1}\right|^2 = \frac{\Re\left[\Lambda\right]}{|\Lambda|^2} \left|\frac{2\Lambda}{\Lambda+1}\right|^2 = \Re\left[\Lambda\right] \left|\frac{2}{\Lambda+1}\right|^2 \\ R_b &= \left|\frac{\Lambda-1}{\Lambda+1}\right|^2 = \left|\frac{1-\Lambda}{1+\Lambda}\right|^2 \end{aligned}$$

However, we might also imagine that there are some absorption at the boundary which breaks this condition which is discussed in [7]. Strictly speaking we should carefully calculate the Poynting vectors (sec.3) in each medium to find the correct formulism (something that I have not done or found a clear derivation in literature for).

6 Modelling Dielectric Media

We are interested in calculating dielectric constants from microscopic models. First, we will discuss the Lorentz Oscillator model, Drude model and others. However in dense media, in order to connect the results of these models to dielectric constants, we must also consider the interactions of many oscillators giving rise to an effective medium as described using the Lorentz-Lorenz (aka. Clausius-Mossotti) relation. We then consider theorems for the effective dielectric constant of mixtures of materials.

6.1 Lorentz Oscillators

This is a classical model of a dielectric material based on the motion of bound electric charges. It's described in many books [8]

Consider a simple model of a bond charge, that of a driven harmonic oscillator:

$$m_0 \frac{\partial^2 x}{\partial t^2} + 2\gamma m_0 \frac{\partial x}{\partial t} + m_0 \omega_0^2 x = qE \tag{109}$$

where the second term accounts for dissipation/absorption. If we look for harmonic (particular) solutions to this equation (we're going to ignore the complementary solutions since they are transient) and we're going to do it using complex notation - $e^{-i\omega t}$- for simplicity then

$$-\omega^2 x_0 - 2\gamma i\omega x_0 + \omega_0^2 x_0 = \frac{q}{m_0} E_0 \tag{110}$$

leading to

$$x_0 = \frac{\frac{q}{m_0} E_0}{\omega_0^2 - \omega^2 - 2i\gamma\omega} \tag{111}$$

Now we can relate this to the polarisation of the medium quite simply using

$$\mathbf{P} = Nq\mathbf{x} \tag{112}$$

where N is the density of dipoles, q is their charge which will be $-e$. If the density is high though, the electric fields from neighbouring dipoles will self-reinforce the polarisation leading to a slightly more complicated relationship; the Clausius-Mossotti relation or Lorentz-Lorenz formula as discussed in sec.6.4. For now we will stick with the simpler relation though. We find that

$$\varepsilon_r = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - 2i\gamma\omega} \tag{113}$$

where ω_p is the plasma frequency for reasons that become clear in sec6.2.

$$\omega_P^2 = \frac{Nq^2}{m_0 \varepsilon_0} \tag{114}$$

Splitting this into real and imaginary parts we get

$$\varepsilon_r' = 1 + \frac{\omega_p^2 \left(\omega_0^2 - \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2} \tag{115}$$

$$\varepsilon_r'' = \frac{2\omega_p^2 \gamma\omega}{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2} \tag{116}$$

Simplification 1 One way these formulae can be simplified by saying that $\omega_0 + \omega \approx 2\omega$. This leads to

$$\varepsilon_r' = 1 + \frac{1}{2\omega} \frac{\omega_p^2 (\omega_0 - \omega)}{(\omega_0 - \omega)^2 + \gamma^2} \tag{117}$$

$$\varepsilon_r'' = \frac{1}{2\omega} \frac{\omega_p^2 \gamma}{(\omega_0 - \omega)^2 + \gamma^2} \tag{118}$$

When n is assumed to be almost constant through the transition, we can use eqn.38

$$\kappa \approx \frac{\varepsilon_r''}{2n'} \tag{119}$$

and eqn.42 to getNow

$$\alpha = \frac{\omega\varepsilon_r''}{cn'} = \frac{1}{2cn'}\frac{\omega_p^2\gamma}{(\omega_0-\omega)^2+\gamma^2} \tag{120}$$

which is the Lorentzian absorption line that we normally expected. Since we're interested in polaritons, we're not going to make this assumption.

Simplification 2 A more sophisticated simplication of the dielectric permittivity can be achieved in another way by defining a new oscillator strength

$$\omega_0'^2 = \omega_0^2 - \gamma^2 \tag{121}$$

and then the dielectric permittivity can be expanded as

$$\varepsilon_r = 1 + \frac{\omega_p^2}{2\omega_0'}\left(\frac{1}{\omega_0'-\omega-i\gamma} + \frac{1}{\omega_0'+\omega+i\gamma}\right) \tag{122}$$

which is equivalent to eqn.113. Now, we can make the 'rotating wave approximation', a fancy way of saying that we neglect the second term because it is small when ω is near ω_0. (Although, we're using complex notation, I think that negative frequencies still don't have any meaning (?)). The permittivity is now

$$\varepsilon_r = 1 + \frac{\omega_p^2}{2\omega_0'}\left(\frac{1}{\omega_0'-\omega-i\gamma}\right) \tag{123}$$

The real and imaginary parts of this relation is

$$\varepsilon_r' = 1 + \frac{\omega_p^2}{2\omega_0'}\left(\frac{\omega_0'-\omega}{(\omega_0'-\omega)^2+\gamma^2}\right) \tag{124}$$

$$\varepsilon_r'' = \frac{\omega_p^2}{2\omega_0'}\left(\frac{\gamma}{(\omega_0'-\omega)^2+\gamma^2}\right) \tag{125}$$

This is almost the same outcome as before but making a cleaner approximation.

In general the complex refractive index is therefore

$$n' = \sqrt{\frac{1}{2} + \frac{\omega_p^2}{4\omega_0'}\left(\frac{\omega_0'-\omega}{(\omega_0'-\omega)^2+\gamma^2}\right) + \frac{\omega_p^2}{4\omega_0'}\frac{\sqrt{\left(\left((\omega_0'-\omega)^2+\gamma^2\right)\frac{2\omega_0}{\omega_p^2}+(\omega_0'-\omega)\right)^2+\gamma^2}}{(\omega_0'-\omega)^2+\gamma^2}} \tag{126}$$

$$\kappa = \sqrt{-\frac{1}{2} - \frac{\omega_p^2}{4\omega_0'}\left(\frac{\omega_0'-\omega}{(\omega_0'-\omega)^2+\gamma^2}\right) + \frac{\omega_p^2}{4\omega_0'}\frac{\sqrt{\left(\left((\omega_0'-\omega)^2+\gamma^2\right)\frac{2\omega_0}{\omega_p^2}+(\omega_0'-\omega)\right)^2+\gamma^2}}{(\omega_0'-\omega)^2+\gamma^2}} \tag{127}$$

And so the absorption is

$$\alpha = \frac{2\omega}{c}\sqrt{-\frac{1}{2} - \frac{\omega_p^2}{4\omega_0'}\left(\frac{\omega_0'-\omega}{(\omega_0'-\omega)^2+\gamma^2}\right) + \frac{\omega_p^2}{4\omega_0'}\frac{\sqrt{\left(\left((\omega_0'-\omega)^2+\gamma^2\right)\frac{2\omega_0}{\omega_p^2}+(\omega_0'-\omega)\right)^2+\gamma^2}}{(\omega_0'-\omega)^2+\gamma^2}} \tag{128}$$

Equations which are somewhat lacking in beauty. A plot of these quantities near the resonance frequency is shown in fig.2.

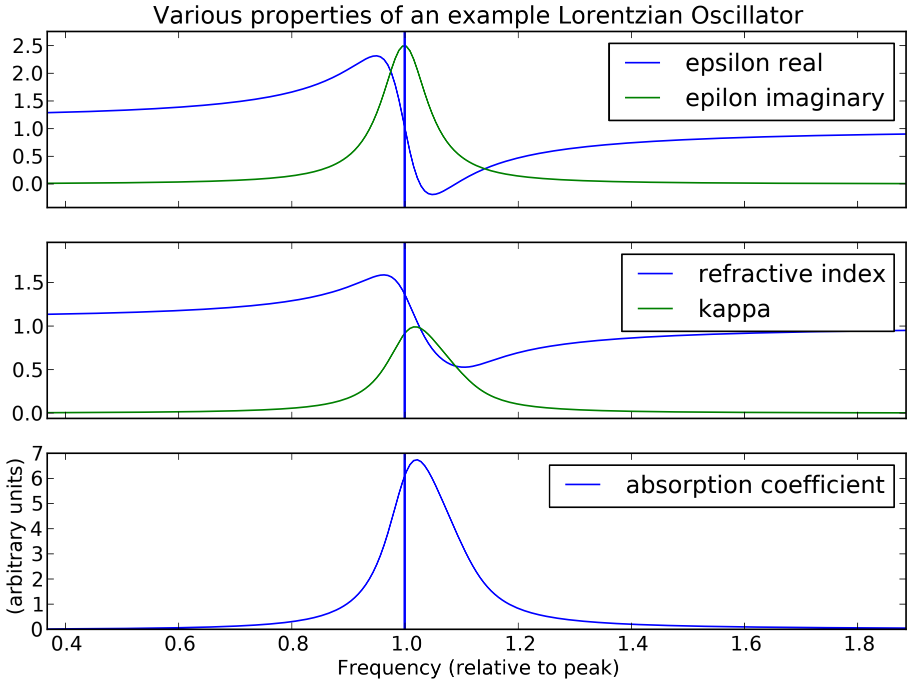


Figure 2: Shows a typical Lorentz oscillator. Notice that the refractive index is tending towards different values either side of the transition. Also the absorption peak isn't in the same place as the original resonance. Although if we wanted to know the absorption of a slab of this material we would need to model it explicitly rather than assume that the resonant absorption is given by the absorption coefficient.

6.1.1 Quantum Effects

The Lorentz model can still be used to describe quantum mechanical transitions through the introduction of a quantity called oscillator strength.

$$f = \frac{2m^*\omega_{12}d_{12}^2}{\hbar e^2} \tag{129}$$

leading to

$$\varepsilon_r = 1 + \frac{\omega_p^2 f}{\omega_0^2 - \omega^2 - 2i\gamma\omega} \tag{130}$$

Here d_{12} is the transition's dipole matrix element and ω_{12} is the (natural) frequency of the transition. Also, in solids, the electron mass is changed by its surrounding bandstructure and so electrons will have an effective mass, $m*$, which replaces m_0 both in the definitions of f and ω_p.

If we include a background dielectric constant, we have

$$\varepsilon_r = \varepsilon_b + \frac{\omega_p^2 f}{\omega_0^2 - \omega^2 - 2i\gamma\omega} \tag{131}$$

where ε_b is the background dielectric constant. However the plasma frequency is often redefined to be

$$\omega_P^2 = \frac{Nq^2}{m^*\varepsilon_r\varepsilon_0} \tag{132}$$

and so we can have

$$\varepsilon_r = \varepsilon_b\left(1 + \frac{\omega_p^2 f}{\omega_0^2 - \omega^2 - 2i\gamma\omega}\right) \tag{133}$$

Often the details of these definitions varies between authors which can be a potential source of confusion. For instance $\omega_p^2 f$ may be redefined as either $\omega_p'^2$ or f'.

6.2 Drude Model - Free Electron Absorption

This models the response of the cloud of charges to an electric field using a classical model of point charges. So

$$m_0 \frac{\partial^2 x}{\partial t^2} + 2\gamma m_0 \frac{\partial x}{\partial t} = qE \tag{134}$$

where the second term accounts for dissipation or damping of the motion via scattering of the electrons etc. We now look for the particular solutions to the differential equation using a harmonic analysis; a.k.a. we assume that both x and E are functions like $e^{-i\omega t}$. Gives

$$-m_0\omega^2 x_0 - 2\gamma m_0 i\omega x_0 = qE_0 \tag{135}$$

so that

$$x_0 = \frac{q}{m_0}\frac{E_0}{-\omega^2 - 2\gamma i\omega} \tag{136}$$

Now we come to a point that is often over-looked. Typically the Drude model is taken to be the Lorentz model with $\omega_0 = 0$ due to the lack of restraining force on the electrons. We can also get to the right result for the Drude model dielectric constant from this point by using $\mathbf{P} = Nq\mathbf{x}$ where N is the density of electric charges. However, doing this leads to confusion when we start looking at local field corrections and this is because we shouldn't use $\mathbf{P} = Nq\mathbf{x}$ for the electron gas. This makes sense because the free-electrons are not well-defined dipoles. Instead we incorporate the effect of the electron gas through their currents. We have

$$\mathbf{J} = Nq\mathbf{v} \tag{137}$$

and eqn.31

$$\mathbf{J} = \sigma\mathbf{E}$$

and so

$$\sigma = \frac{Nq^2}{m_0}\frac{i\omega}{\omega^2 + 2\gamma i\omega} \tag{138}$$

The DC conductivity is given by

$$\sigma_0 = \frac{Nq^2}{m_0 2\gamma} = \frac{Nq^2\tau}{m_0} \tag{139}$$

where τ is the scattering time of the electron. Then we have[6]

$$\sigma = \frac{\sigma_0}{1 - i\omega\tau}$$

Alternatively, using eqn.34, we get

$$\varepsilon_r = 1 - \frac{\omega_p^2}{\omega^2 + 2\gamma i\omega} \tag{140}$$

where the 'plasma frequency' is defined

$$\omega_P^2 = \frac{Nq^2}{m_0\varepsilon_0} \tag{141}$$

It can be shown that only waves with frequencies higher than ω_p will propagate in this medium. Otherwise the dielectric constant will be largely imaginary and so the medium will be highly reflective.

We finally note that quantum theory shows that electrons possess an effective mass $m*$ in a semiconductor, which replaces m_0 in the above equations.

[6]with the usual problems of minus signs between physicist and engineering conventions.

6.2.1 Metals

Even at terahertz frequencies, it is often acceptable to use the dc conductivity to define the dielectric constant, i.e.

$$\varepsilon_r = \varepsilon_b + \mathrm{i}\frac{\sigma_0}{\varepsilon_0 \omega} \tag{142}$$

The complex refractive index is then

$$n = \left(\sqrt{\frac{\varepsilon_b + \sqrt{\varepsilon_b^2 + \left(\frac{\sigma_0}{\varepsilon_0\omega}\right)^2}}{2}} + i\sqrt{\frac{-\varepsilon_b + \sqrt{\varepsilon_b^2 + \left(\frac{\sigma_0}{\varepsilon_0\omega}\right)^2}}{2}} \right) \tag{143}$$

but since the conductivity is much bigger than the dielectric constant this becomes

$$n = (1+i)\sqrt{\frac{\sigma_0}{2\varepsilon_0\omega}} \tag{144}$$

At higher frequencies, the Drude model becomes more useful for describing the dielectric constant and in particular the electron scattering rate can be important. Then in the optical spectral range, metals begin to reach their plasma frequencies and then interband absorptions become important [9].

Thin metal layers tend to have a lower conductivity than a bulb sample due to a higher number of imperfections [10]. There is also a temperature dependence to the refractive index dependent on whether the electron scattering rate is dominated by defects or phonons[10].

A common extension to the Drude-Model is to use a frequency dependent scattering rate (?).

6.3 Other Models

The Drude and Lorentz models are very useful especially given their relative simplicity. Slightly more advanced versions of the theories may be known by other names. For instance, including a background dielectric constant in the Drude model might strictly be called the modified Drude model or the Drude-Sommerfield model. Combining a Lorentz dielectric and a Drude model can give good empirical fits to data for metals at optical frequencies and these might be called Lorentz-Drude models. But in fact, nomenclature varies; Lorentz-Drude may also refer to what I have termed the Drude model and Lorentz-Summerfield may be used for models that use the electron effective mass or quantum mechanical models of scattering rates.

There are many other models for dielectric constants. For permenantly polarised media there is the Debye relaxation model which is useful to describe the dielectric constant of water.

$$\varepsilon(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 - i\omega\tau_D} \tag{145}$$

There is also the more general Havriliak-Negami relaxation

$$\varepsilon(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{\left(1 + (-i\omega\tau_D)^\beta\right)^\gamma} \tag{146}$$

and special cases, Cole-Davidson relaxation $\varepsilon(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{(1-i\omega\tau_D)^\beta}$ and Cole-Cole relaxation $\varepsilon(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1+(-i\omega\tau_D)^\beta}$. There is also the Maxwell-Wagner-Sillars effect which describes a capacitor-like effect that occurs at internal phase boundaries.

For composite media, there are the theories of effective media such as Maxwell-Garnett, Bruggeman and many more which are described in sec.6.5.

Finally, we haven't even begun to discuss quantum mechanical models which are necessary for more accurate descriptions of a materials dielectric constant [ref that article].

6.4 Local field corrections: Lorentz-Lorenz or Clausius-Mossotti Relations

If the density of dipoles in a dielectric is high, the electric fields from neighbouring dipoles will self-reinforce the polarisation leading to a slightly more complicated relationship between an individual oscillator's susceptibility and the dielectric constant (aka. the permittivity); this is known as the Clausius-Mossotti relation and also as the Lorentz-Lorenz relation. This is covered well in [8, 11, 12]

$$N\chi_0 = 3\frac{n^2-1}{n^2+2} \tag{147}$$

equivalently

$$N\chi_0 = 3\frac{\varepsilon_r-1}{\varepsilon_r+2} \tag{148}$$

where χ_0 is the susceptibility of a single charge. I believe that this also applies to lossy materials but I have not proven it.

Kittel puts

$$N\varepsilon_0\chi_0 = 3\frac{\varepsilon_r-1}{\varepsilon_r+2} \tag{149}$$

but I think that it is wrong (according to my definitions at least).

The proof first shows that a dipole feels a local field given by

$$\mathbf{E}_{local} = \mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0} \tag{150}$$

This works well in a cubic crystal but other crystal structures are perhaps different. Then

$$\mathbf{P} = \varepsilon_0 N\chi_0 \mathbf{E}_{local} \tag{151}$$

and

$$\varepsilon_0\left(\varepsilon_r-1\right)\mathbf{E} = \mathbf{P} \tag{152}$$

which can easily be solved to give eqn.148. Noting that without this correction, we have $\varepsilon_r = 1 + N\chi_0$, the correction becomes

$$\varepsilon_r' = 1 + \frac{3\varepsilon_r-3}{4-\varepsilon_r} = \frac{2\varepsilon_r+1}{4-\varepsilon_r} \tag{153}$$

In [12], Feynman argues that a free electron gas does not feel local electric field corrections due to the other electrons because they do not each produce an electric field. This is clearer if we consider that the effects of the electron gas should be described using a current/conductivity term rather than a polarisation. Still I am not clear and to model a material where the bound and free charges might interact.

6.5 Effective Media: Maxwell-Garnett, Bruggeman etc.

These are more advanced models for calculating the effective refractive index / dielectric constant of agragate materials. In fact there is a whole field of effective media models which is beyond the scope of these notes.

6.6 Thin Dielectric Layers

It can be useful to replace a stack of dielectric layers with an effective dielectric constant. This is a kind of effective medium and is described in various places [11, 13, 5]; Born and Wolf call this form birefringence as the resulting effective medium will be uniaxial even when all of the layers are dielectric.

For the electric field parallel to the interfaces, we get

$$\varepsilon_{xx} = \sum_i f_i\varepsilon_i \tag{154}$$

where f_i is the fractional volume/width of each layer and ε_i is its dielectric constant.

Whereas for electric field perpendicular to the interfaces (along the growth direction), the dielectric constant is given by

$$\frac{1}{\varepsilon_{zz}} = \sum_i \frac{f_i}{\varepsilon_i} \tag{155}$$

6.7 Modelling Gold

As an example of modelling real data, in the article by Etchegoin [9] the dielectric constant of gold was modelled using a combination of two transitions and a Drude model. However, the absorptions are not quite simple Lorentzians and this changes the absorption line-profile. The final function is

$$\varepsilon(\omega) = \varepsilon_\infty + \frac{\omega_p^2}{\omega^2 + i\Gamma\omega} + G_1(\omega) + +G_2(\omega) \tag{156}$$

$$G_i(\omega) = C_i\left[\frac{\exp\left(-i\frac{\pi}{4}\right)}{\omega_i - \omega - i\Gamma_i} + \frac{\exp\left(i\frac{\pi}{4}\right)}{\omega_i + \omega + i\Gamma_i}\right] \tag{157}$$

where $\varepsilon_\infty = 1.53$, $\omega_p = 12.9907\,\mathrm{PHz}$, $\Gamma = 0.110803\,\mathrm{PHz}$, $C_1 = 3.78340\,\mathrm{PHz}$, $\omega_1 = 4.024897\,\mathrm{PHz}$, $\Gamma_1 = 0.81898\,\mathrm{PHz}$, $C_2 = 7.73947\,\mathrm{PHz}$, $\omega_2 = 5.69079\,\mathrm{PHz}$, $\Gamma_2 = 2.00388\,\mathrm{PHz}$. Note that these frequencies are all natural units rather than real units (factor of 2π difference).

Moreover, the paper suggests that the following function is useful in general for describing real data

$$G_i(\omega) = C_i\left[e^{i\phi_i}\left(\omega_i - \omega - i\Gamma_i\right)^{\mu_i} + e^{-i\phi_i}\left(\omega_i + \omega + i\Gamma_i\right)^{\mu_i}\right] \tag{158}$$

which they call a critical point transition (see also [14]).

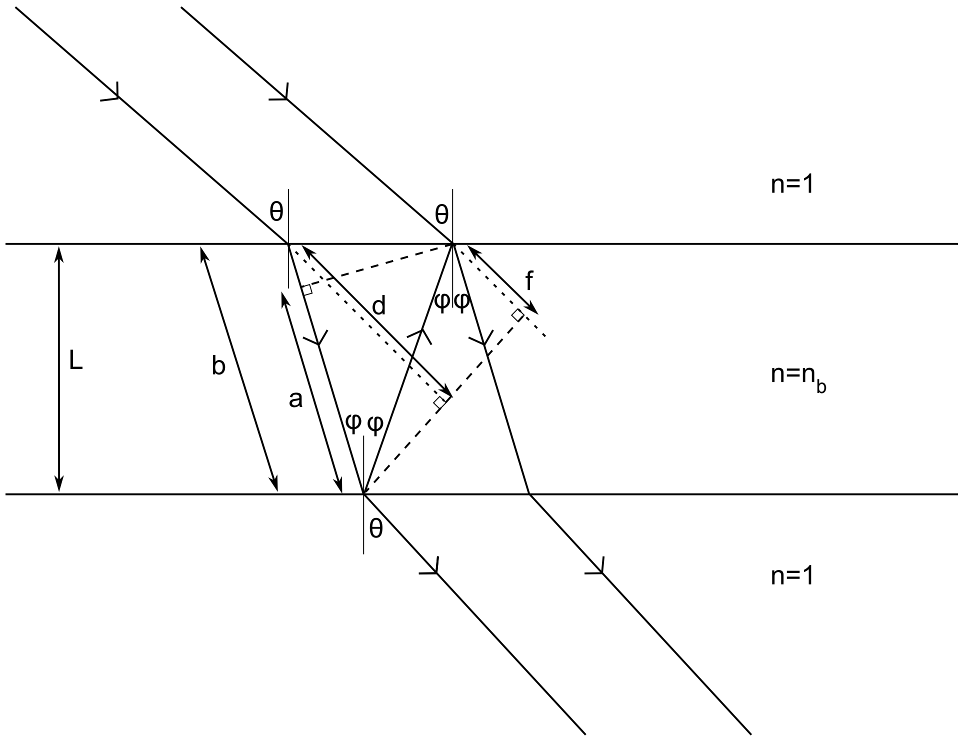


Figure 3: A ray passing through a simple slab.

Part II
Waves in Layered Media

7 An Etalon

An etalon is any optical component with two highly parallel interfaces, so it could be a plane/slab of dielectric or it could be two parallel mirrors (although this is also known as a Fabry Perot). The reflections of the light between the two interfaces interfere such that the transmittance and reflectance of the layer is sensitive to wavelength, angle and polarisation. There are two approaches to modelling the problem, either we sum over all of the possible multiple reflections within the structure or we look for a steady state solution that implicitly performs that summation for us.

7.1 Phase changes across sheet

I always get the phase changes wrong when I first try to do this calculation after a break. It's important to include the effect of the sideways shift and the phase shift of the beam that would have occured if the the slab had not been present.

Phase Change Top to Bottom

$$\phi_1 = k_b b - k_0 d \tag{159}$$

where k_0 is the initial wavevector in vacuum and k_b is the wavevector in the slab. b and d are distances.

$$b = \frac{L}{\cos\varphi} \tag{160}$$

$$d = b\cos(\theta - \varphi) \tag{161}$$

Leads to

$$\begin{aligned}\phi_1 &= \frac{k_b L}{\cos\varphi}\left(1 - \frac{\cos(\theta-\varphi)}{n_b}\right) \\ &= k_b L\left(\cos\varphi - \frac{\cos\theta}{n_b}\right) \end{aligned} \tag{162}$$

Phase Change Bottom to Top

$$\phi_2 = k_b b + k_0 f \tag{163}$$

$$f = b\cos(\theta+\varphi) \tag{164}$$

leads to

$$\begin{aligned}\phi_2 &= \frac{k_b L}{\cos\varphi}\left(1+\frac{\cos(\theta+\varphi)}{n_b}\right)\\ &= k_b L\left(\cos\varphi - \frac{\cos\theta}{n_b}\right)\end{aligned} \tag{165}$$

Phase Change For One Internal Reflection

$$\phi_1 + \phi_2 = 2k_b L\cos\varphi \tag{166}$$

Generally, this phase shift is all that matters and is derived from

$$\phi_1 + \phi_2 = k_b b + k_b a \tag{167}$$

$$a = b\cos(2\varphi) \tag{168}$$

Leads to

$$\phi_1 + \phi_2 = \frac{k_b L}{\cos\varphi}\left(1+\cos(2\varphi)\right) = 2k_b L\cos\varphi \tag{169}$$

Consequently, in most books and derivations, ϕ_1 is normally defined simply as

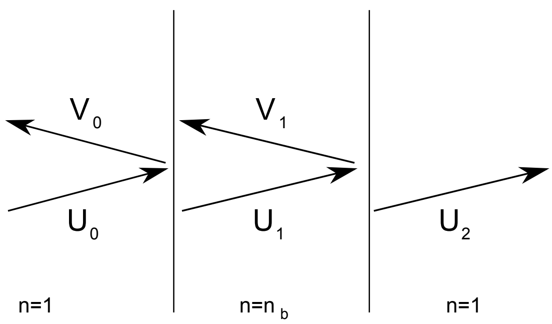
$$\phi_1 = k_b L\cos\varphi \tag{170}$$

which can be done because the extra term simplifies to

$$\frac{\omega}{c}L\cos\theta \tag{171}$$

which is independent of the properties of the dielectric layer (ω is a constant across the layers).

Reflection and Transmission of Etalon To solve the reflection and transmission of the slab, we use a steady state model i.e. we make sure that the waves in each section of the problem balance at the interfaces.



$$E_r = r_{01}E_0 + t_{10}E_b e^{i\phi_2} \tag{172}$$

$$E_a = t_{01}E_0 + r_{10}E_b e^{i\phi_2} \tag{173}$$

$$E_b = r_{12}E_a e^{i\phi_1} \tag{174}$$

$$E_t = t_{12}E_a e^{i\phi 1} \tag{175}$$

We find

$$E_a = \frac{t_{01}}{1 - r_{10}r_{12}e^{i\phi_1 + i\phi_2}}E_0 \tag{176}$$

and so

$$E_t = \frac{t_{12}t_{01}e^{i\phi_1}}{1 - r_{10}r_{12}e^{i\phi_1 + i\phi_2}} E_0 \tag{177}$$

$$E_r = \left(r_{01} + \frac{r_{12}t_{10}t_{01}e^{i\phi_1 + i\phi_2}}{1 - r_{10}r_{12}e^{i\phi_1 + i\phi_2}} \right) E_0 \tag{178}$$

Using the following useful relations

$$r_{01} = -r_{10} \tag{179}$$

and

$$t_{01}t_{10} - r_{01}r_{10} = 1 \tag{180}$$

which are true in general; we may rewrite these equations as

$$E_t = \frac{t_{01}t_{12}e^{i\phi_1}}{1 + r_{01}r_{12}e^{i\phi_1 + i\phi_2}} E_0 \tag{181}$$

$$E_r = \left(\frac{r_{01} + r_{12}e^{i\phi_1 + i\phi_2}}{1 + r_{01}r_{12}e^{i\phi_1 + i\phi_2}} \right) E_0 \tag{182}$$

This being something that people like to do.

8 Transfer Matrix

The transfer matrix method allows us to calculate the reflection and transmission of a stack of dielectric layers (also called thin films). It is covered in many books and papers (these are just a few [4, 5, 15, 1, 12, 3, 7, 16, 17, 6, 18]) and is not only used in optics but also in quantum mechanics and r.f. waveguide etc.

There seem to be at least three types of derivation. One for each of the following types of vectors

$$\begin{pmatrix} E_L \\ E_R \end{pmatrix} \quad \begin{pmatrix} E \\ H \end{pmatrix} \quad \begin{pmatrix} E \\ \frac{\partial E}{\partial z} \end{pmatrix}$$

8.1 Snell's Law in a dielectric stack

First we note that Snell's law holds between all pairs of levels in the stack.

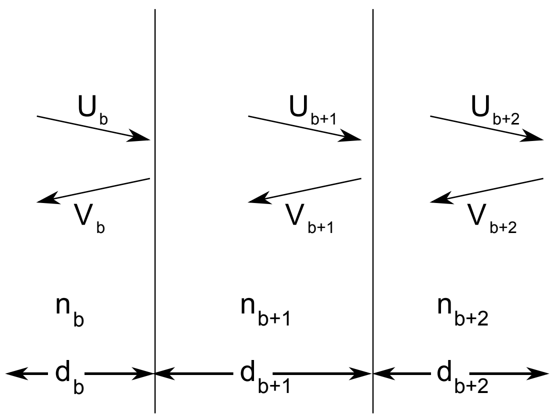
$$n_0 \sin\theta_0 = n_1 \sin\theta_1 = n_2 \sin\theta_2 = n_3 \sin\theta_3 = \ldots = \alpha \tag{183}$$

and also that the cosine may be calculated using.

$$\cos\theta_b = \sqrt{1 - \left(\frac{\alpha}{n_b}\right)^2} \tag{184}$$

These relations are often used to reduce the number of trigonometric calculations within a program.

8.2 Derivation (i): Left and right traveling waves



This formulism is very useful as it also crops up in various other fields too, transmission line problems, quantum heterostructures, optical fibres Useful references are [4, 1, 7, 3, 6, 18, 17]. If we define the origin of the waves in each layer as just to the left of each interface, then the waves in the next layer (just to the right of the interface) are

$$\begin{pmatrix} e^{-i\delta_{b+1}}U_{b+1} \\ e^{i\delta_{b+1}}V_{b+1} \end{pmatrix} \tag{185}$$

where

$$\delta_{b+1} = k_{b+1}d_{b+1}\cos\left(\theta_{b+1}\right) \tag{186}$$

Then we can write the following equations

$$\begin{aligned} V_b &= r_{b,b+1}U_b + t_{b+1,b}e^{i\delta}V_{b+1} & (187) \\ e^{-i\delta}U_{b+1} &= t_{b,b+1}U_b + r_{b+1,b}e^{i\delta}V_{b+1} & (188) \end{aligned}$$

These can be rearranged to give

$$\begin{aligned} U_b &= \frac{1}{t_{b,b+1}}\left(e^{-i\delta}U_{i+1} - r_{b+1,b}e^{i\delta}V_{b+1}\right) & (189) \\ V_b &= \frac{1}{t_{b,b+1}}\left(r_{b,b+1}e^{-i\delta}U_{i+1} + \left(t_{b,b+1}t_{b+1,b} - r_{b,b+1}r_{b+1,b}\right)e^{i\delta}V_{b+1}\right) & (190) \end{aligned}$$

Now we can use the following useful relations

$$r_{b+1,b} = -r_{b,b+1} \tag{191}$$

and

$$t_{b,b+1}t_{b+1,b} - r_{b,b+1}r_{b+1,b} = 1 \tag{192}$$

Some sources say that the above relations don't hold for absorbing media but I can't see why this would be the case. Others don't use the relations in order to model unusual surfaces or the effects or surface roughness.

we can write

$$\begin{pmatrix} U_b \\ V_b \end{pmatrix} = \frac{1}{t_{b,b+1}}\begin{pmatrix} e^{-i\delta} & r_{b,b+1}e^{i\delta} \\ r_{b,b+1}e^{-i\delta} & e^{i\delta} \end{pmatrix}\begin{pmatrix} U_{b+1} \\ V_{b+1} \end{pmatrix} \tag{193}$$

or we can define separate matrices for the interfaces and the layers i.e.

$$\begin{pmatrix} U_b \\ V_b \end{pmatrix} = \frac{1}{t_{b,b+1}}\begin{pmatrix} 1 & r_{b,b+1} \\ r_{b,b+1} & 1 \end{pmatrix}\begin{pmatrix} e^{-i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix}\begin{pmatrix} U_{b+1} \\ V_{b+1} \end{pmatrix} \tag{194}$$

Typically, our exit layer will be defined slightly differently as it will be semi-infinite (i.e. just the interface is counted).

$$\frac{1}{t_{b,b+1}}\begin{pmatrix} 1 & r_{b,b+1} \\ r_{b,b+1} & 1 \end{pmatrix} \tag{195}$$

Then in a normal problem, we want to find the reflection and transmission of the system. i.e. we set $V_{-1} = 0$ and then find $V_0$and U_{-1}as functions of U_0. So

$$\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} U_{-1} \\ 0 \end{pmatrix} \tag{196}$$

leading to

$$\begin{aligned} t &= \frac{1}{A} & (197) \\ r &= \frac{C}{A} & (198) \end{aligned}$$

In a real program, we can substitute simplified versions of the Fresnel equations, ie.

$$\begin{aligned} r &= \frac{1-\Lambda}{1+\Lambda} & (199) \\ t_s &= \frac{2}{1+\Lambda} & (200) \\ t_p &= \frac{2}{1+\Lambda}\frac{n_1}{n_2} & (201) \end{aligned}$$

(from sec.4) and in fact, we can note that the extra term of $\frac{n_1}{n_2}$ in the p-pol case will cancel between layers except for the layers at the extremes of the structure. If we are willing to redefine the transmissivity as

$$T = \Lambda_{0-1} |t|^2 \tag{202}$$

then we can drop this extra fraction entirely! Although the p-pol transmission coefficients will strictly speaking be wrong by this factor. Here Λ_{0-1} is Λ evalutated from the initial and final layers. The matrix for a layer would be

$$\frac{1}{2} \begin{pmatrix} (1+\Lambda) e^{-i\delta} & (1-\Lambda) e^{i\delta} \\ (1-\Lambda) e^{-i\delta} & (1+\Lambda) e^{i\delta} \end{pmatrix} \tag{203}$$

for both polarisations (remembering that Λ is polarisation sensitive).

8.3 Derivation (ii): E and H fields

I will just summarise from Born and Wolf [5] who rigorously prove an electromagnetic theory of stratisfied media, see also [15].

For TE waves, we have

$$E_x = U(z) e^{i(k_0 \alpha y - \omega t)} \tag{204}$$
$$H_y = V(z) e^{i(k_0 \alpha y - \omega t)} \tag{205}$$
$$H_z = W(z) e^{i(k_0 \alpha y - \omega t)} \tag{206}$$
$$p = \sqrt{\frac{\varepsilon}{\mu}} \cos\theta \tag{207}$$

For TM waves, we have

$$H_x = U(z) e^{i(k_0 \alpha y - \omega t)} \tag{208}$$
$$E_y = -V(z) e^{i(k_0 \alpha y - \omega t)} \tag{209}$$
$$E_z = -W(z) e^{i(k_0 \alpha y - \omega t)} \tag{210}$$
$$p = \sqrt{\frac{\mu}{\varepsilon}} \cos\theta \tag{211}$$

where $\alpha = n \sin\theta$ and n , ε, μ and θ are z-dependent. We also have the relation

$$\alpha U + \mu W = 0 \tag{212}$$

They show that within a stack, U and V can be calculated using a characteristic matrix

$$\mathbf{Q}_0 = \mathbf{M}_1(z_1)\mathbf{M}_2(z_2 - z_1)\mathbf{M}_3(z_3 - z_2) \ldots \mathbf{M}_b(z_b - z_{b-1})\mathbf{Q}_b \tag{213}$$

where

$$\mathbf{Q}_b = \begin{pmatrix} U_b(z) \\ V_b(z) \end{pmatrix} \tag{214}$$

and the characteristic matrix is given by

$$\mathbf{M}_b = \begin{pmatrix} \cos(k_0 n_b z \cos\theta_b) & -\frac{i}{p} \sin(k_0 n_b z \cos\theta_b) \\ -ip \sin(k_0 n_b z \cos\theta_b) & \cos(k_0 n_b z \cos\theta_b) \end{pmatrix} \tag{215}$$

where I've added some subscripts just to improve the clarity. From the characteristic matrix, it can easily be seen that we normally replace z with the thickness of each layer. In Ch.13, Born and Wolf indicate that this formulism also works for lossy media as long as ε and k (and θ) are allowed to be complex.

In order to calculate transmission and reflection with this formulism, we must expand U_0 and V_0 in terms of the incident and reflected waves, so that we have

$$\begin{pmatrix} I + rI \\ p_0 (I - rI) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} tI \\ p_l tI \end{pmatrix} \tag{216}$$

this leads to

$$r = \frac{(A + Bp_l)\, p_0 - (C + Dp_l)}{(A + Bp_l)\, p_0 + (C + Dp_l)} \tag{217}$$

and

$$t = \frac{2p_0}{(A + Bp_l)\, p_0 + (C + Dp_l)} \tag{218}$$

where

$$R = |r|^2 \quad \text{and} \quad T = \frac{p_l}{p_0} |t|^2 \tag{219}$$

Here, I note a slight deviousness that has occurred. As shown in sec.4, compared to the TE or s-pol case, t_p doesn't quite simplify when using $p = \sqrt{\frac{\mu}{\varepsilon}} \cos\theta$ as there is an extra $\frac{n_1}{n_2}$ present. Here Born & Wolf have redefined the geometrical factor in the relation between transmittance and transmission coefficient to sneakily account for this term. However this means that the transmission *coefficient* is subtly wrong for TM light.

This formulism is less concerned with interfaces than with layers. As an example, a single layer between two semi-infinite media only requires one matrix in this case. In the previous formulism we would need to multiply two matrices together, one for each interface.

8.4 Derivation (iii): E and spatial derivative

Missing

9 Properties of the transfer matrix

We will considering only the first derivation (sec.8.2). The transfer matrix defines the system through

$$\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} U_f \\ V_f \end{pmatrix}$$

for the electric fields. Then we can calculate the system's properties.

$$t = \left.\frac{U_f}{U_0}\right|_{V_f=0} = \frac{1}{A} \tag{220}$$

$$t_b = \left.\frac{V_0}{V_f}\right|_{U_0=0} = \frac{DA - BC}{A} = \frac{\Lambda_{0f}}{A} \tag{221}$$

$$r = \left.\frac{V_0}{U_0}\right|_{V_f=0} = \frac{C}{A} \tag{222}$$

$$r_b = \left.\frac{U_f}{V_f}\right|_{U_0=0} = -\frac{B}{A} \tag{223}$$

where t_b and r_b are transmission and reflectivity for a wave entering the back of the filter. We could simplify t_b since it involved the determinant of the matrix. It is a property of a square matrix multiplication that

$$\det(\mathbf{XYZ}...) = \det(\mathbf{X}) \det(\mathbf{Y}) \det(\mathbf{Z}) ...$$

therefore consider the two types of matrix, firstly, interface matrices are given by

$$\mathbf{A} = \frac{1}{t_{b,b+1}} \begin{pmatrix} 1 & r_{b,b+1} \\ r_{b,b+1} & 1 \end{pmatrix}$$

we can see that their determinants are

$$\det(\mathbf{A}) = \frac{1 - r_{b,b+1}^2}{t_{b,b+1}^2} = \Lambda_{b,b+1} \tag{224}$$

and secondly, for a layer matrices, we have

$$\mathbf{L} = \begin{pmatrix} e^{-i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix}$$

so

$$\det(\mathbf{L}) = 1 \tag{225}$$

Therefore the determinant of the whole transfer matrix must be

$$\det(\mathbf{M}) = \Lambda_{0f} \tag{226}$$

and so we can write the backwards transmission of the filter as

$$t_b = \frac{\Lambda_{0f}}{A}$$

This result also shows that the thin film filter has the same transmission coefficient in both directions since

$$T = \Lambda_{0f}\frac{1}{|A|^2}$$

and

$$T_b = \Lambda_{f0}\left|\frac{\Lambda_{0f}}{A}\right|^2 = \Lambda_{0f}^*\frac{1}{|A|^2}$$

since $\Lambda_{f0} = 1/\Lambda_{of}$ and so assuming that the initial and final mediums are non-absorbing

$$T_b = T$$

and this result holds even if the filter itself is absorbing. Although the transmission must be the same (a result that can also be argued thermodynamically), the reflection coefficients are able to be different if the filter is absorbing.

In summary, we can write the Transmission and Reflectivities as

$$T = T_b = \Re\left[\Lambda_{0f}\right]\frac{1}{|A|^2} \tag{227}$$

$$R = \frac{|C|^2}{|A|^2} \tag{228}$$

$$R_b = \frac{|B|^2}{|A|^2} \tag{229}$$

See also [19]

10 Electric field within a Thin Film Stack

It's relatively straight forward to find the electric field profile within a thin film stack. Using the first derivation for the transfer matrix, we can easily see that the values of the fields just to the left of each boundary are given by

$$\begin{pmatrix} U_j \\ V_j \end{pmatrix} = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}\begin{pmatrix} U_{-1} \\ 0 \end{pmatrix} \tag{230}$$

where we use the transfer matrix from layer j to the end of the stack. We know already know that

$$U_{-1} = \frac{U_0}{A} \tag{231}$$

where A is the component from the total transfer matrix. Therefore

$$U_j = \frac{A_j}{A}U_0 \tag{232}$$

$$V_j = \frac{C_j}{A}U_0 \tag{233}$$

We can now write the electric field inside the layer. For the s-polarisation (TE polarisation), this is

$$E_y^{(j)} = U_j \exp\left(-ik_z^{(j)}\Delta z\right) + V_j \exp\left(ik_z^{(j)}\Delta z\right) \tag{234}$$

where we keep to the sign definitions that we used in the Fresnel equations section and Δz is the distance from the back interface of the layer. Likewise, for the p- / TM polarisation, we write

$$\begin{aligned} E_x^{(j)} &= \left(U_j \exp\left(-ik_z^{(j)}\Delta z\right) - V_j \exp\left(ik_z^{(j)}\Delta z\right)\right)\cos\theta_j & (235)\\ E_z^{(j)} &= \left(U_j \exp\left(-ik_z^{(j)}\Delta z\right) + V_j \exp\left(ik_z^{(j)}\Delta z\right)\right)\sin\theta_j & (236)\end{aligned}$$

where θ_j is the angle of the electric field to the plane of the layers (in an isotropic layer this is also the angle of the k-vector to the stack normal). Again, there are some potential issues with signs depending on the definitions of the Fresnel coefficients used in the transfer matrix calculation.

There is one additional consideration which is that there is a term of $\frac{n_i}{n_j}$ that is dropped from the calculation of each interface matrix in 'TM' polarisation (see sec.8.2). This is because these terms cancel out except for the first and last refractive indices. This final extra term is then re-included in the calculation by redefining the relation between the Transmittance and the transmission coefficient, however, we can not do this when we want to know the electric field strengths within the structure and so we must include the extra factor.

Generally, we should expect E_x to be smoothly varying since the dielectric boundary conditions show that E_x is continuous across interfaces. On the other hand, it is D_z which is continous across an interface and so E_z should show abrupt jumps between the layers. Plotting $\varepsilon_{zz}E_z$ (D_z) might be useful for checking the consistency of our calculations.

10.1 Absorbing layers

When the layer is absorbing, we have accepted that we can use complex angles to describe our Fresnel coefficients and the phase shift across the layer, this is generally accepted to be correct. However, what is the actual electrical field within the layer? Since, we have not carried out the derivation, it is hard to be certain. Ideally, we would solve the Fresnel coefficients for an interface between a dielectric and a lossy medium, by considering the possible values of inhomogeneous waves in the lossy medium. Then we would compare our solution to the complex angle solution. Further we should find the possible wave solutions for a bounded thin layer of lossy material. Ideally, we would find a reference to some work that has already done this for us!

For now we will assume that the electric field can still be described using the previously defined equations even in a lossy material.

11 Modelling Absorption within the layer stack

Once we know the electric fields within the structure, we can work out where absorption is taking place. This is useful for designing photodetectors or photovoltaic structures. In [6], the (relative) absorption is derived from a consideration of the Poynting vector and calculated using

$$dA = q_j F(z) dz \tag{237}$$

where $F(z) = |E(z)|^2 / |E_0(z)|^2$ which for p-pol is

$$F(z) = \left(|E_x(z)|^2 + |E_z(z)|^2\right) / \left|E_{0p}(z)\right|^2 \tag{238}$$

, for s-pol is

$$F(z) = \left|E_y(z)\right|^2 / |E_{0s}(z)|^2 \tag{239}$$

and for unpolarised light is

$$F(z) = \frac{1}{2}\left(|E_x(z)|^2 + |E_z(z)|^2 + \left|E_y(z)\right|^2\right) / \left(\left|E_{0p}(z)\right|^2 + |E_{0s}(z)|^2\right) \tag{240}$$

q_j is then a coefficient describing the layer, given by

$$q_j = Re\left(\frac{n_j \cos\theta_j}{n_0 \cos\theta_0}\right) 2Im\left(k_z^{(j)}\right) \tag{241}$$

Since the first term is the one used to convert field amplitudes to power ratios and the second term describes the absorption along the z direction, we can see that the equation is very similar to $dI = \alpha I dz$ where $\alpha = {}^{2\omega n''}/_{c}$. When the zeroth layer is non-absorbing, then this can be written as

$$q_j = 2\frac{n'_j n''_j}{n_0 \cos\theta_0}\frac{\omega}{c} \tag{242}$$

since

$$Re\left(n_j \cos\theta_j\right) Im\left(n_j \cos\theta_j\right) = n'_j n''_j \tag{243}$$

.

Proof :

$$\cos\theta_j = \sqrt{1-\sin^2\theta_j} = \sqrt{1-\frac{n_0^2 \sin^2\theta_0}{n_j^2}}$$

putting $\cos\theta_j = z'_j + iz''_j$, we can write

$$\begin{aligned} z_j'^2 - z_j''^2 + 2iz'_j iz''_j &= 1-\frac{n_0^2\sin^2\theta_0}{n_j^2} \\ &= 1-\frac{n_0^2\sin^2\theta_0}{\left(n_j'^2-n_j''^2\right)^2+\left(2n'_j in''_j\right)^2}\left(n_j'^2-n_j''^2-2in'_j in''_j\right) \end{aligned}$$

therefore

$$\begin{aligned} Re\left(n_j\cos\theta_j\right) Im\left(n_j\cos\theta_j\right) &= \left(n'_j z'_j - n''_j z''_j\right)\left(n'_j z''_j + n''_j z'_j\right) \\ &= n'_j n''_j\left(z_j'^2 - z_j''^2\right) + \left(n_j'^2 - n_j''^2\right) z'_j z''_j \\ &= n'_j n''_j \end{aligned}$$

In [17], the absorption is given as

$$\begin{aligned} Q(z) &= \frac{1}{2}c\varepsilon_0\frac{4\pi n''_j}{\lambda}n'_j\left|E(z)\right|^2 && (244) \\ &= \varepsilon_0\omega n''_j n'_j\left|E(z)\right|^2 && (245) \end{aligned}$$

where Q is the average power disipated at z and λ is the vacuum wavelength. This is similar to above but seems to assume normal incidence. There is also a factor difference of initial irradiance which is given by

$$I = \frac{c\varepsilon_0 n_0}{2}\left|E_0(z)\right|^2 \tag{246}$$

where ε_0 is the vacuum dielectric constant but n_0 is the refractive index of the initial layer.

12 Incoherent Transfer Matrix

If the layers are very thick then we can no longer assume that the phases of the traversing beams are mutually coherent and so, we should work with intensities instead of electric fields. (Various theories exist in the literature to account for intermediate cases.) The resulting theory is almost identical to the first version of the transfer matrix theory.

We can write

$$I_b^- = R_{b,b+1} I_b^+ + T_{b+1,b} e^{-\alpha_{b+1}} I_{b+1}^- \tag{247}$$
$$e^{\alpha_{b+1}} I_{b+1}^+ = R_{b+1,b} e^{-\alpha_{b+1}} I_{b+1}^- + T_{b,b+1} I_b^+ \tag{248}$$

where $\alpha = \frac{2}{c}\omega\Im\left[n\cos\theta\right] d$ [7]

$$\begin{pmatrix} I_b^+ \\ I_b^- \end{pmatrix} = \frac{1}{T_{b,b+1}} \begin{pmatrix} 1 & -R_{b+1,b} \\ R_{b,b+1} & T_{b+1,b}T_{b,b+1} - R_{b+1,b}R_{b,b+1} \end{pmatrix} \begin{pmatrix} e^{\alpha_{b+1}} & 0 \\ 0 & e^{-\alpha_{b+1}} \end{pmatrix} \begin{pmatrix} I_{b+1}^+ \\ I_{b+1}^- \end{pmatrix} \tag{249}$$

In fact the first matrix is the general matrix for any object in this system i.e.

$$\frac{1}{T_{b,b+1}} \begin{pmatrix} 1 & -R_{b+1,b} \\ R_{b,b+1} & T_{b+1,b}T_{b,b+1} - R_{b+1,b}R_{b,b+1} \end{pmatrix} \tag{250}$$

For instance, we can see that to describe an absorbing medium (without its interfaces), we will have $T_{b,b+1} = T_{b,b+1} = e^{-\alpha_{b+1}}$ and $R_{b,b+1} = R_{b+1,b} = 0$, which will give the second matrix in the transfer matrix expression above. For another example, a typical dielectric interface between two media will have $T_{b,b+1} = T_{b+1,b}$, $R_{b,b+1} = R_{b+1,b}$ and $T_{b,b+1} + R_{b,b+1} = 1$, giving us

$$\frac{1}{T_{b,b+1}} \begin{pmatrix} 1 & -R_{b,b+1} \\ R_{b,b+1} & 1 - 2R_{b,b+1} \end{pmatrix} \tag{251}$$

In exactly the same way as we do for the coherent system, we can find the transmittance and reflectance for the incoherent system

$$\begin{pmatrix} I_0^+ \\ I_0^- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_f^+ \\ 0 \end{pmatrix} \tag{252}$$

so that

$$\frac{I_f^+}{I_0^+} = T = \frac{1}{A} \tag{253}$$
$$\frac{I_0^-}{I_0^+} = R = \frac{C}{A} \tag{254}$$

We will also have

$$\frac{I_0^-}{I_f^-} = T_b = \frac{DA - BC}{A} = \frac{1}{A} \tag{255}$$
$$\frac{I_f^+}{I_f^-} = R_b = -\frac{B}{A} \tag{256}$$

The determinant of the system is the product of the determinants of the constituent matrices which we can show is given by

$$\det(\mathbf{M}) = \frac{1}{T_{b,b+1}^2}\left(T_{b+1,b}T_{b,b+1} - R_{b+1,b}R_{b,b+1} + R_{b+1,b}R_{b,b+1}\right) = \frac{T_{b+1,b}}{T_{b,b+1}} = 1$$

since the transmission is the same in both directions and so the total system must have a determinant of unity.

[7] For the uniaxial layers of partIII, it will be $\alpha = \frac{2}{c}\omega\Im\left[n_{xx}\sqrt{1 - \frac{n_0^2}{n_{zz}^2}\sin^2\theta_0}\right] d$

12.1 An Important Issue for Absorbing Media

There is a thorny problem surrounding the definition of T for absorbing layers that it is usually possible to ignore. Normally we are able to define

$$T = \left(\frac{n_2 \cos\theta_t}{n_1 \cos\theta_i}\right) |t|^2$$

or using more usually

$$T = \Lambda \left|\frac{2}{1+\Lambda}\right|^2$$

where

$$\Lambda_s = \frac{n_2 \cos\theta_2}{n_1 \cos\theta_1} \qquad \text{or} \qquad \Lambda_p = \frac{n_1 \cos\theta_2}{n_2 \cos\theta_1}$$

but this gives us a complex transmissivity when either or both media are absorbing. That doesn't seem physical and moreover, it means that we will no-longer have conservation of energy at the interface. Since we expect that

$$R + T = 1$$

when media are absorbing (and so Λ are complex), then the obvious correction to make is to define

$$T = \Re\left[\Lambda\right] \left|\frac{2}{1+\Lambda}\right|^2 \tag{257}$$

since

$$\left|\frac{1-\Lambda}{1+\Lambda}\right|^2 + \Re\left[\Lambda\right] \left|\frac{2}{1+\Lambda}\right|^2 = 1 \tag{258}$$

This implies that we must define the obliqueness factor differently for the TE / s and TM /p polarisations. s-polarisation is defined as usual

$$T_s = \Re\left(\frac{n_2 \cos\theta_t}{n_1 \cos\theta_i}\right) |t_s|^2$$

but p-polarisation (TM) is now defined as

$$T_p = \Re\left(\frac{n_2^* \cos\theta_t}{n_1^* \cos\theta_i}\right) \left|t_p\right|^2$$

This is the approach taken by Ohta in [6] and should work when either or both media are absorbing assuming that we are justified in assuming that $R + T = 1$. However I have also seen definitions where [ref]

$$T = \frac{\Re\left[n_2 \cos\theta_t\right]}{\Re\left[n_1 \cos\theta_i\right]} |t|^2$$

which are not equivalent since in general $\Re\left[\frac{z_2}{z_1}\right] \neq \frac{\Re[z_2]}{\Re[z_1]}$.

Using our new definition, we can write the matrix for an interface using

$$\frac{1}{4\Re\left[\Lambda\right]} \begin{pmatrix} |1+\Lambda|^2 & -|1-\Lambda|^2 \\ |1-\Lambda|^2 & |1+\Lambda|^2 - 2|1-\Lambda|^2 \end{pmatrix}$$

where we note that the factor of $1/\Re\left[\Lambda\right]$ might no-longer cancel between layers of the calculation for absorbing layers and so we should keep it within the calculation since in general $\Re\left[\frac{1}{z}\right] \neq \frac{1}{\Re[z]}$

12.2 Incorporating a thin film structure

If we want to incorporate a thin film filter into an largely incoherent system then we need to work out the transmissivity and reflectivity of the filter, so we can fill in the terms of the generic incoherent matrix

$$\frac{1}{T_{b,b+1}} \begin{pmatrix} 1 & -R_{b+1,b} \\ R_{b,b+1} & T_{b+1,b}T_{b,b+1} - R_{b+1,b}R_{b,b+1} \end{pmatrix}$$

The thin film system matrix of the thin film is

$$\begin{pmatrix} U_b \\ V_b \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} U_{b+1} \\ V_{b+1} \end{pmatrix} \tag{259}$$

for the electric fields. We can calculate the system's properties using the results in sec.9

$$\begin{aligned} T_{b,b+1} &= T_{b,b+1} = & \Re\left[\Lambda_{if}\right] \frac{1}{|A|^2} \\ R_{b,b+1} &= & \frac{|C|^2}{|A|^2} \\ R_{b+1,b} &= & \frac{|B|^2}{|A|^2} \end{aligned}$$

The fourth term of the incoherent matrix is the most complicated

$$T_{b+1,b}T_{b,b+1} - R_{b+1,b}R_{b,b+1} = \frac{1}{|A|^4}\left(\Re\left[\Lambda_{if}\right]^2 - |C|^2|B|^2\right) \tag{260}$$

so the thin film can be described within an incoherent system using the matrix

$$\frac{1}{\Re\left[\Lambda_{if}\right]} \begin{pmatrix} |A|^2 & -|B|^2 \\ |C|^2 & \frac{\Re\left[\Lambda_{if}\right]^2 - |C|^2|B|^2}{|A|^2} \end{pmatrix} \tag{261}$$

where we note again that the factor of $1/\Re\left[\Lambda\right]$ might no-longer cancel between layers of the calculation when there are absorbing layers

12.3 Power Density vs Depth

It is likely that we can approximately find the power density within the structure using...?

12.4 Alternative Approach to Incoherent Layers

Instead of using the incoherent transfer matrix formulism, in [18] another approach is suggested. Here the coherent transfer matrix equations are used but thick layers have an additional randomness added to their phase shift to simulate incoherence. The calculation must be run several times and the results averaged in order to find the expected result. Hence

$$\begin{pmatrix} e^{-i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix} \rightarrow \begin{pmatrix} e^{-i\delta+i\psi} & 0 \\ 0 & e^{i\delta+i\psi'} \end{pmatrix} \tag{262}$$

where $\psi = \beta \cdot Rand$ with $Rand$ a uniformly randomised number between -1 to 1 and β as a scaling factor between 0 and π. This approach has the additional advantage that it can be used to simulate partial coherence. It can also be used to check the results of the fully incoherent transfer matrix approach, both for the reflectances/transmittances and the intensity distribution within the structure. To model a totally incoherent layer we should set $\beta = \pi$ and in the paper the calculation was run 30 times (with different random numbers) and the results averaged. The result was then smoothed using a filter of 10 moving averages, in order to reduce the noise. This gave a curve approximately the same as the incoherent transfer matrix approach.

Part III
Uniaxial Layers

Anisotropic media exhibit new phenomena: Double refraction, extraordianary/ordianary waves, k not parallel to direction of motion. However, In this part, we will *only* be concerned with a type of uniaxial layer which has its extraordinary axis along along the growth direction of the stack of layers (perpendicular to the plane of the layers).

13 Wave Equation: derivation 3 - Anisotropic Media

Considering $\nabla\times$eqn.3 leads to

$$\underline{\nabla}\left(\underline{\nabla}.\mathbf{E}\right)-\nabla^2\mathbf{E}=-\frac{\partial\left(\underline{\nabla}\times\mathbf{B}\right)}{\partial t} \tag{263}$$

We can not eliminate the first term as we have done before. Using eqn.4 with eqn.14 and assuming that there are no free currents, the right hand side of the equation becomes a second time derivative of $\mathbf{D}$

$$-\underline{\nabla}\left(\underline{\nabla}.\mathbf{E}\right)+\nabla^2\mathbf{E}=\mu_0\frac{\partial^2\mathbf{D}}{\partial t^2} \tag{264}$$

Assuming that $\mathbf{D}=\varepsilon_0\underline{\underline{\varepsilon_r}}*\mathbf{E}$ where the dielectric constant is a rank 2 tensor with frequency dependence, we can still as usual take $\underline{\underline{\varepsilon_r}}$ outside of the time derivative

$$-\underline{\nabla}\left(\underline{\nabla}.\mathbf{E}\right)+\nabla^2\mathbf{E}=\mu_0\varepsilon_0\underline{\underline{\varepsilon_r}}*\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{265}$$

If we look for plane wave solutions, we come to

$$-\mathbf{k}\left(\mathbf{k}.\mathbf{E}\right)+k^2\mathbf{E}=\omega\mu_0\varepsilon_0\underline{\underline{\varepsilon_r}}*\mathbf{E} \tag{266}$$

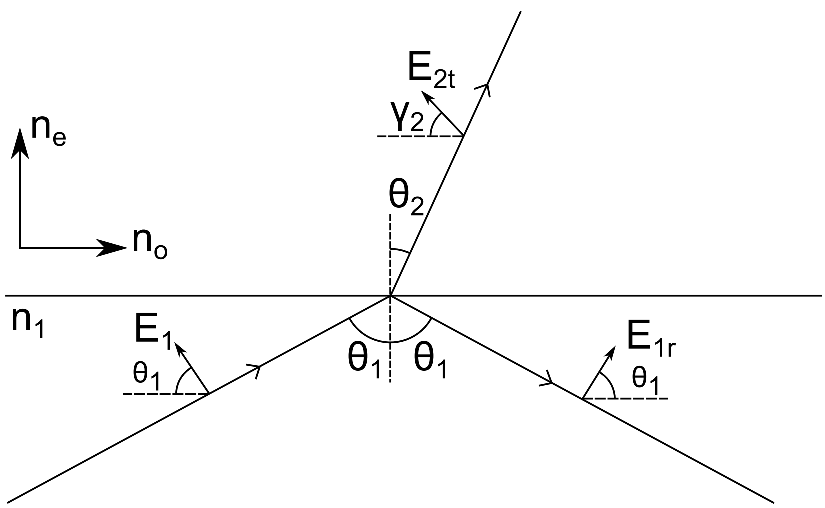
Likewise, $\nabla\times$eqn.4 and following through as before leads to

$$\nabla^2\mathbf{B}=\mu_0\frac{\partial\left(\underline{\nabla}\times\mathbf{D}\right)}{\partial t} \tag{267}$$

it is unclear how to simplify $\nabla\times\left(\underline{\underline{\varepsilon_r}}*\mathbf{E}\right)$ or alternatively $\nabla\times\left(\underline{\underline{\varepsilon_r}}\mathbf{E}\right)$. In an homogeneous medium $\underline{\varepsilon_r}$ won't have any spatial dependence.

14 Deriving the Fresnel Coefficients for out particular type of uniaxial layer

Since we are only be concerned with uniaxial layers which have their optical axis perpendicular to the interface, we can immediately state the Fresnel coefficients for waves polarised perpendicular to the plane of incidence (s-polarisation or TE polarisation) will be the standard coefficients for isotropic media (using ordinary refractive index for the uniaxial medium). Therefore the following derivation is concerned only with the in-plane polarised case (p-polarisation or TM polarisation).



We can initially follow the standard path for the derivation of the Fresnel coefficients except that the angle of the Electric field to the plane of the interface is not the same as the angle of the wave's k-vector to the interface's normal. The figure shows this more simply than words can explain it. The relation between the two angles is given by

$$\tan\gamma = \frac{\varepsilon_o}{\varepsilon_e}\tan\theta \tag{268}$$

see sec.18. So by applying that the two boundary condions

$$\begin{aligned} E_{1\parallel} &= E_{2\parallel} && (269)\\ D_{1\perp} &= D_{2\perp} && (270)\end{aligned}$$

we can quickly reach the following equations

$$\begin{aligned} (E_1 - E_{1r})\cos\theta_1 &= E_{2t}\cos\gamma_2 && (271)\\ \varepsilon_1\,(E_1 + E_{1r})\sin\theta_1 &= \varepsilon_{2e}E_{2t}\sin\gamma_2 && (272)\\ \frac{\omega}{c}n_1\sin\theta_1 = k_{1\parallel} &= k_{2\parallel} && (273)\end{aligned}$$

where ε_{2e} is the extraordinary refractive index of the layer (while ε_{2o} is the ordinary refractive index) and the last equation normally leads to Snell's law. So we can solve this to get

$$t_p = \frac{E_{2t}}{E_1} = \frac{2\cos\theta_1\sin\theta_1\varepsilon_1}{\varepsilon_1\sin\theta_1\cos\gamma_2 + \varepsilon_{2e}\sin\gamma_2\cos\theta_1} \tag{274}$$

but we would like to simplify this equation, let's try

$$t_p = \frac{1}{\cos\gamma_2}\frac{2\cos\theta_1\sin\theta_1\varepsilon_1}{\varepsilon_1\sin\theta_1 + \varepsilon_{2o}\tan\theta_2\cos\theta_1} \tag{275}$$

where I have substituted for the relation between γ_2 and θ_2. We can see geometrically that

$$\tan\theta_2 = \frac{k_{2\parallel}}{k_{2\perp}} \tag{276}$$

so we see that

$$\tan\theta_2 = \frac{k_{1\parallel}}{k_{2\perp}} = \frac{\omega}{c}\frac{n_1\sin\theta_1}{k_{2\perp}} \tag{277}$$

we also know the plane wave solution to uniaxial layer's wave equation

$$\begin{aligned} \frac{k_\perp^2}{\varepsilon_o} + \frac{k_\parallel^2}{\varepsilon_e} &= \left(\frac{\omega}{c}\right)^2 && (278)\\ \Rightarrow k_{2\perp} &= n_{2o}\frac{\omega}{c}\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1} && (279)\end{aligned}$$

so

$$\tan\theta_2 = \frac{n_1\sin\theta_1}{n_{2o}\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1}} \tag{280}$$

So now we can write

$$t_p = \frac{\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1}}{\cos\gamma_2}\frac{2\cos\theta_1 n_1}{n_1\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1} + n_{2o}\cos\theta_1} \tag{281}$$

Q. Is $\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1} \equiv \cos\theta_2$?

What about the factor at the front?

$$\frac{\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1}}{\cos\gamma_2} = \sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1}\sqrt{1 + \left(\frac{\varepsilon_{2o}}{\varepsilon_{2e}}\tan\theta\right)^2}$$

$$= \sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}} \sin^2\theta_1 + \left(\frac{\varepsilon_{2o}}{\varepsilon_{2e}}\right)^2 \frac{\varepsilon_1 \sin^2\theta_1}{\varepsilon_{2o}}} = \sqrt{1 - \left(1 - \frac{\varepsilon_{2o}}{\varepsilon_{2e}}\right) \frac{\varepsilon_1 \sin^2\theta_1}{\varepsilon_{2e}}} \tag{282}$$

Note that this factor is almost 1 as long as $\varepsilon_{2o} \approx \varepsilon_{2e}$. Therefore the transmission is

$$t_p = \sqrt{1 - \left(1 - \frac{\varepsilon_{2o}}{\varepsilon_{2e}}\right) \frac{\varepsilon_1 \sin^2\theta_1}{\varepsilon_{2e}}} \frac{2\cos\theta_1 n_1}{n_1\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1} + n_{2o}\cos\theta_1} \tag{283}$$

We will now find the reflection coefficient, we can write

$$\begin{aligned}
(E_1 - E_{1r}) \frac{\cos\theta_1}{\cos\gamma_2} &= \frac{\varepsilon_1}{\varepsilon_{2e}} (E_1 + E_{1r}) \frac{\sin\theta_1}{\sin\gamma_2} && (284) \\
r_p = \frac{E_{1r}}{E_1} &= \frac{\varepsilon_{2e}\sin\gamma_2\cos\theta_1 - \varepsilon_1\sin\theta_1\cos\gamma_2}{\varepsilon_{2e}\sin\gamma_2\cos\theta_1 + \varepsilon_1\sin\theta_1\cos\gamma_2} && (285)
\end{aligned}$$

as before, we use the fact that

$$\tan\gamma_2 = \frac{n_{2o}}{\varepsilon_{2e}} \frac{n_1\sin\theta_1}{\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1}} \tag{286}$$

to derive

$$r_p = \frac{n_{2o}\cos\theta_1 - n_1\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1}}{n_{2o}\cos\theta_1 + n_1\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1}} \tag{287}$$

which we see doesn't have any strange factors at the front. Now we will consider the twin problem of a wave in our uniaxial layer impinging upon an interface with a dielectric medium. Because, the optical axis of the uniaxial layer is perpendicular to the interface, we can assume that the reflection angle is still given by the angle of incidence and we can quickly write down our boundary equations as before:

$$\begin{aligned}
(E_1 - E_{1r})\cos\gamma_1 &= E_{2t}\cos\theta_2 && (288) \\
\varepsilon_{1e}(E_1 + E_{1r})\sin\gamma_1 &= \varepsilon_2 E_{2t}\sin\theta_2 && (289) \\
k_{1\|} &= k_{2\|} = \frac{\omega}{c} n_2 \sin\theta_2 && (290)
\end{aligned}$$

so we can easily find

$$t'_p = \frac{2\varepsilon_{1e}\sin\gamma_1\cos\gamma_1}{\varepsilon_2\sin\theta_2\cos\gamma_1 + \varepsilon_{1e}\sin\gamma_1\cos\theta_2} \tag{291}$$

$$r'_p = \frac{\varepsilon_2\sin\theta_2\cos\gamma_1 - \varepsilon_{1e}\sin\gamma_1\cos\theta_2}{\varepsilon_2\sin\theta_2\cos\gamma_1 + \varepsilon_{1e}\sin\gamma_1\cos\theta_2} \tag{292}$$

as before, we can use

$$\tan\gamma_1 = \frac{n_{1o}}{\varepsilon_{1e}} \frac{n_2\sin\theta_2}{\sqrt{1 - \frac{\varepsilon_2}{\varepsilon_{1e}}\sin^2\theta_2}} \tag{293}$$

to derive

$$t'_p = \frac{1}{\sqrt{1 - \left(1 - \frac{\varepsilon_{2o}}{\varepsilon_{2e}}\right)\frac{\varepsilon_1\sin^2\theta_1}{\varepsilon_{2e}}}} \frac{2n_{1o}\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1}}{n_2\sqrt{1 - \frac{\varepsilon_2}{\varepsilon_{1e}}\sin^2\theta_2} + n_{1o}\cos\theta_2} \tag{294}$$

$$r'_p = \frac{n_2\sqrt{1 - \frac{\varepsilon_2}{\varepsilon_{1e}}\sin^2\theta_2} - n_{1o}\cos\theta_2}{n_2\sqrt{1 - \frac{\varepsilon_2}{\varepsilon_{1e}}\sin^2\theta_2} + n_{1o}\cos\theta_2} \tag{295}$$

we can see that we have the inverse factor to the first case and this makes it clear why the factor is dropped from transfer matrix calculations, since it cancels out due to the form of the interface matrices. However, is we want to find the electric field in each layer, then we will need to remember to include it!

15 An Etalon of this particular case of uniaxial material

Amazingly we can use the normal equation for an etalon as dervied in sec.7, as long as we use the Fresnel coefficients derived in the previous section. So

$$t = \frac{t_{12}t_{23}e^{i\delta_2}}{1 + r_{12}r_{23}e^{2i\delta_2}} \tag{296}$$

We don't even need to include the extra factors for the transmissivities since they cancel out for the product $t_{01}t_{12}$. Therefore, we only need to substitute

$$\cos\theta_2 \rightarrow \sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1}$$

for all of the terms of the equation. So for example

$$\delta_2 = \sqrt{1 - \frac{\varepsilon_1}{\varepsilon_{2e}}\sin^2\theta_1}\frac{n_{2o}\omega d}{c}$$

In sec.23 we simplify this equation for a special case related to quantum well intersubband transitions.

16 Transfer Matrix with General Anisotropic Media

Berreman[20] and Yeh[21, 22, 23, 24] have derrived matrix methods of dealing with arbitrarily orientated biaxial layers. These are based upon 4x4 matrices since the TE and TM polarisations are no longer independent in such media. I believe that these may also describe gyroscopic, Faraday and magneto-optical effects. Here are some references on this type of theory [25, 26, 27, 28, 29, 30, 31, 32]

17 Transfer Matrix for a particular case of uniaxial material.

For studying intersubband absorption in quantum wells, we have a lossy uniaxial material with optical axis along the growth direction. After a little consideration, we can see that the TM waves will be extraordinary waves while TE waves will be ordinary. This can be described with a 2x2 matrix method [13]. I shall summarise this method here.

The wave equation for anisotropic medium is given by

$$k^2\mathbf{E} - \mathbf{k}\left(\mathbf{k}.\mathbf{E}\right) = K^2\underline{\underline{\varepsilon_r}}\mathbf{E} \tag{297}$$

(see sec.13) where $K = \frac{\omega}{c}$, one can find wavevectors of the form $\mathbf{k} = [k_x, 0, k_z]$ where

$$k_z^2 = \varepsilon_{xx}K^2 - \frac{\varepsilon_{xx}}{\varepsilon_{zz}}k_x^2 \tag{298}$$

It is also stated that $k_x(k_{\shortparallel})$ is still continuous across interfaces (is this always true? for any anisotropic medium?).

In [13], almost the same transfer matrix formulism is used as was previously derived in sec.8.2 for dielectric medium. The matrix for an anisotropic layer is given as

$$\begin{pmatrix} e^{-i\delta_j} & 0 \\ 0 & e^{i\delta_j} \end{pmatrix} \tag{299}$$

where $\delta_j = k_{z_j}d_j$ and k_{z_j} is the value found from eqn.298. The matrix describing interfaces with anisotropic layers is then given to be

$$\begin{pmatrix} I_{ij}^{+} & I_{ij}^{-} \\ I_{ij}^{-} & I_{ij}^{+} \end{pmatrix} \tag{300}$$

where

$$I_{ij}^{\pm} = \frac{k_z^{(i)}\varepsilon_j \pm k_z^{(j)}\varepsilon_i}{2k_z^{(i)}\varepsilon_j} \tag{301}$$

where $\varepsilon_j = \varepsilon_{xx}$ for an anisotropic layer. This is only valid for the TM/p-polarised case (see the k-vector form of the fresnel equations in sec.4), there is strictly speaking a factor of $\frac{n_2}{n_1}$ missing from the matrix and an additional factor for the uniaxial layers (sec.14) but these cancel between interfaces and so we can just consider $\frac{n_f}{n_0}$ (in addition to the normal geometric factor for transmissivity), so that we have finally

$$T = \left(\frac{k_z^{(f)} \varepsilon_0}{k_z^{(0)} \varepsilon_f} \right) |t|^2 \tag{302}$$

(assuming that the initial and final media are not uniaxial) where we calculate t from the system's transfer matrix

$$\mathbf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{303}$$

from which

$$t = \frac{1}{A} \tag{304}$$

$$r = \frac{C}{A} \tag{305}$$

In the TE/s-polarised case, the equations are

$$I_{ij}^{\pm} = \frac{k_z^{(i)} \pm k_z^{(j)}}{2k_z^{(i)}} \tag{306}$$

and

$$T = \left(\frac{k_z^{(f)}}{k_z^{(0)}} \right) |t|^2 \tag{307}$$

17.1 Compatability of previous formulism with isotropic transfer matrix

If we look at the version of the Fresnel equations given in sec.4.6, we see that we can rewrite matrix 300 as

$$\frac{1}{t_{ij}} \begin{pmatrix} 1 & r_{ij} \\ r_{ij} & 1 \end{pmatrix} \tag{308}$$

(ignoring the extra factor of $\frac{n_2}{n_1}$ in the the TM case as discussed before). Therefore, we can easily incorporate the uniaxial layer into an otherwise isotropic transfer matrix calculation as long as we calculate the correct values for r, t and δ (the phase shift). Normally, we might calculate the transmission amplitudes as

$$r = \frac{1 - \Lambda}{1 + \Lambda} \tag{309}$$

$$t = \frac{2}{1 + \Lambda} \tag{310}$$

where for TE polarisation, we use

$$\Lambda_{TE} = \frac{n_j \cos\theta_j}{n_i \cos\theta_i} \tag{311}$$

and the TM polarisation, we use

$$\Lambda_{TM} = \frac{n_i \cos\theta_j}{n_j \cos\theta_i} \tag{312}$$

However, we could equally define Λ as

$$\Lambda_{TE} = \frac{k_z^{(j)}}{k_z^{(i)}} \tag{313}$$

for TE-pol and

$$\Lambda_{TM} = \frac{\varepsilon_1 k_z^{(j)}}{\varepsilon_2 k_z^{(i)}} \tag{314}$$

for TM-pol. Since in an isotropic medium we have $k_z = \frac{\omega}{c} n \cos\theta$, we can easily see that these forms are identical. Therefore, we only need to replace our definition for $\cos\theta$ for the uniaxial layers in order to model them correctly in an otherwise 'standard' transfer matrix calculation! For the uniaxial layers, we should use

$$\cos\theta \to \sqrt{1 - \frac{\varepsilon_0 \sin^2\theta_0}{\varepsilon_{zz}}} \tag{315}$$

where $\varepsilon_0 \sin^2\theta_0$ is constant for all layers in the thin film structure and that's it.

18 The Electric Field in Uniaxial Layers and Absorbing Layers

Section 10 seems quite straight forward but we should our special uniaxial layers too. For our uniaxial material, we have another issue since the electric field is no longer perpendicular to the k-vector. In fact, we can show that ([15] eqn 15.11)

$$\frac{E_z}{E_x} = \pm \frac{\varepsilon_{xx}}{\varepsilon_{zz}} \frac{k_x}{k_z} \tag{316}$$

$$\Rightarrow \tan\gamma = \pm \frac{\varepsilon_{xx}}{\varepsilon_{zz}} \tan\theta \tag{317}$$

where γ is the new angle of the electric field vectors wrt. the plane of the layers and so the electric field inside the layer is given by

$$E_x^{(j)} = \left(U_j \exp\left(-ik_z^{(j)}\Delta z\right) - V_j \exp\left(ik_z^{(j)}\Delta z\right)\right) \cos\gamma_j \tag{318}$$

$$E_z^{(j)} = \left(U_j \exp\left(-ik_z^{(j)}\Delta z\right) + V_j \exp\left(ik_z^{(j)}\Delta z\right)\right) \sin\gamma_j \tag{319}$$

However, there is an additional factor in the transmission coefficient of the uniaxial layer of

$$\sqrt{1 - \left(1 - \frac{\varepsilon_{jxx}}{\varepsilon_{jzz}}\right) \frac{\varepsilon_1 \sin^2\theta_1}{\varepsilon_{jzz}}} \equiv \frac{\sqrt{1 - \frac{1}{\varepsilon_{jzz}} \varepsilon_1 \sin^2\theta_1}}{\cos\gamma_j} \tag{320}$$

this factor is removed when the waves leave the layer and so can be dropped when only performing the transfer matrix calculations of the total reflectance or transmittance. However, this extra factors means that the electric field is

$$E_x^{(j)} = \left(U_j \exp\left(-ik_z^{(j)}\Delta z\right) - V_j \exp\left(ik_z^{(j)}\Delta z\right)\right) \cos\theta_j \tag{321}$$

$$E_z^{(j)} = \left(U_j \exp\left(-ik_z^{(j)}\Delta z\right) + V_j \exp\left(ik_z^{(j)}\Delta z\right)\right) \frac{n_{jxx}}{n_{jzz}} \sin\theta_j \tag{322}$$

where we understand $\cos\theta_j \equiv \sqrt{1 - \frac{1}{\varepsilon_{jzz}} \varepsilon_1 \sin^2\theta_1}$ and $\sin\theta_j = \frac{n_1}{n_{jzz}} \sin\theta_1$

19 Absorptivity in Uniaxial Layers

From section 11, it seems likely that the absorption of an uniaxial layer that only interacts for electric field along the z-direction, can be found from

$$dA = Re\left(\frac{n_j \cos\theta_j}{n_0 \cos\theta_0}\right) 2Im\left(k_z^{(j)}\right) F_z(z) dz \tag{323}$$

where $F_z(z) = |E_z(z)|^2 / |E_{0p}(z)|^2$ and we calculate $\cos\theta_j$ and $k_z^{(j)}$ using the method's discussed in sec.17 and 17.1 i.e.

$$\cos\theta_j = \sqrt{1 - \frac{\varepsilon_0 \sin^2\theta_0}{\varepsilon_{zz}}} \tag{324}$$

$$k_z^{(j)} = n_{xx} \frac{\omega}{c} \cos\theta_j \tag{325}$$

where $n_{xx}^2 = \varepsilon_{xx}$, $n_{zz}^2 = \varepsilon_{zz}$.. so putting $\cos\theta_j = z_j' + iz_j''$, we can write

$$\begin{aligned} z_j'^2 - z_j''^2 + 2iz_j' iz_j'' &= 1 - \frac{n_0^2 \sin^2\theta_0}{\varepsilon_{zz}} \\ &= 1 - \frac{n_0^2 \sin^2\theta_0}{\left(\varepsilon_{zz}'\right)^2 + \left(\varepsilon_{zz}''\right)^2}\varepsilon_{zz}^* \end{aligned} \tag{326}$$

as before

$$\begin{aligned} Re\left(n_{xxj}\cos\theta_j\right) Im\left(n_{xxj}\cos\theta_j\right) &= n_{xxj}' n_{xxj}'' \left(z_j'^2 - z_j''^2\right) + \left(n_{xxj}'^2 - n_{xxj}''^2\right) z_j' z_j'' \\ &= n_{xxj}' n_{xxj}'' + \left\{\left(n_{xxj}'^2 - n_{xxj}''^2\right)\frac{\varepsilon_{zz}''}{2} - n_{xxj}' n_{xxj}'' \varepsilon_{zz}'\right\} \frac{n_0^2 \sin^2\theta_0}{\left(\varepsilon_{zz}'\right)^2 + \left(\varepsilon_{zz}''\right)^2} \end{aligned} \tag{327}$$

and if $n_{xxj}'' = 0$, then we can write

$$Re\left(n_{xxj}\cos\theta_j\right) Im\left(n_{xxj}\cos\theta_j\right) = n_{xxj}'^2 \frac{\varepsilon_{zz}''}{2} \frac{n_0^2 \sin^2\theta_0}{\left(\varepsilon_{zz}'\right)^2 + \left(\varepsilon_{zz}''\right)^2} \tag{328}$$

20 Incoherent transfer matrix for this case of uniaxial layers

Issues with the obliqueness factor? What is the definition of the transmission coefficient for these unixial layers?

Part IV

Quantum Well Intersubband Transitions

21 Introduction

In this case, quantum wells are semi-conductor structures consisting of nanometre thick layers of material chosen so that electrons are trapped within the layers. The layers are so thin that the electron behaviour is governed by quantum mechanics [8]. Such structures are described/introduced in many books on solid state physics and many places on the web, also see [16, 36, 33, 37]. A semiconductor has conduction bands with electrons and valence bands with holes; for many devices such as diode lasers we are interested in the interband transitions between the valence and conduction bands. However, there are also intersubband transitions (ISBTs) within each band, between the different quantised levels of the quantum wells (QWs). These transitions are anisotropic and in particular, they are uniaxial and can only interact with electric field perpendicular to the plane of the well. Hence, they are well described by the theory of Part III (what a surprise!).

22 Simplest Approach

Quantum well transitions are sharply defined in frequency and so are well described by Lorentz oscillators (sec.6.1), although there are some subleties due to the transition's anisotropic nature. Hence, the first approach is to consider the ISBT as a thin slab of dielectric. The absorption of the transition is given approximately by

$$\alpha_{ISBT}L_{QW}=\frac{n}{2c}\frac{\sin^2\theta}{\cos\theta}\frac{\omega_p^2 f_{12}\gamma}{\left(\omega_{12}-\omega\right)^2+\gamma^2}L_{QW} \tag{329}$$

where f_{12} is the oscillator strength, ω_p is the plasma frequency, ω_{12} is the transition frequency, γ is the transition broadening, n is the refractive index, c is the speed of light and θ is the angle of incidence on the well. Since the ISBTs only interact with electric field normal to the plane of the well, the $\sin^2\theta$ term accounts for the angle between the electric field vector and the well normal, hence there will be no absorption at normal incidence. The $\cos\theta$ term accounts for the effective change in thickness of the well with angle or equivalently, the change in electron density per beam cross-section. The plasma frequency is defined as

$$\omega_p^2=\frac{e^2N}{m^*\varepsilon\varepsilon_0}=\frac{e^2N_{2D}}{m^*\varepsilon\varepsilon_0 L_{QW}} \tag{330}$$

where N is the electron density, N_{2D} is the 2d density and L_{QW} is the thickness of the quantum well. m^* is the effective mass of the conduction band electrons (for instance for GaAs, $m^*=0.067m_e$). The oscillator strength is

$$f_{12}=\frac{2m^*\omega_{12}d_{12}^2}{\hbar e^2} \tag{331}$$

where d_{12} is the dipole matrix element between the two quantum states ($d_{12}=\langle 1\left|z\right|2\rangle$).

22.1 Complications - depolarisation shift

The simple approach given above is not always sufficient. One additional phenomena is that there is a shift to the frequency of the transition called the *depolarisation shift*[33, 34, 35]. The dielectric is not thick enough to have a normal Lorentz field and so we don't employ the Clausius-Mossotti or Lorentz-Lorenz relations (sec.6.4). However, the polarisation of the layer does lead to surface charge densities (see sec.6.6). These create a *depolarisation* field that change the effective field seen by the ISBTs through

$$\varepsilon_b E_{\perp 0}=\varepsilon_{well}\left(1+\chi_{ISBT}\right)E_{\perp well} \tag{332}$$

[8] see my other project pyQW and its notes.

which is just the normal electric field change for any dielectric interface (The electric field parallel to the interface is just given by $E_{\|0} = E_{\|well}$ as normal). The argument then goes that the absorption is proportional to the ac current of the layer

$$j_{\perp} = \sigma_{\perp} E_{\perp well} \tag{333}$$

where the conductivity is given through

$$\varepsilon_{ISBT} = \varepsilon_{well}\left(1 + \chi_{ISBT}\right) = \varepsilon_{well}\left(1 + i\frac{\sigma_{\perp}}{\omega\varepsilon_{well}\varepsilon_0 L_{eff}}\right)$$

$$\sigma_{\perp} = -i\omega\varepsilon_{well}\varepsilon_0 L_{eff}\chi_{ISBT} \tag{334}$$

where L_{eff} is the (effective) thickness of the transition layer. Instead we can define a modified conductivity through

$$j_{\perp} = \tilde{\sigma}_{\perp} E_{\perp 0} \tag{335}$$

where

$$\tilde{\sigma}_{\perp} = -i\omega\varepsilon_b\varepsilon_0 L_{eff}\frac{\chi_{ISBT}}{1 + \chi_{ISBT}} \tag{336}$$

$$= -i\omega\varepsilon_b\varepsilon_0 L_{eff}\left(1 - \frac{1}{1 + \chi_{ISBT}}\right) \tag{337}$$

Therefore the absorption is dependent upon the inverse of the transition's dielectric constant! We can see that if we substitute the susceptibility for a Lorentzian oscillator into this formula, we get

$$1 - \frac{1}{1 + \chi_{ISBT}} = \frac{\omega_p^2 f_{12}}{\omega_0^2 + \omega_p^2 f_{12} - \omega^2 - 2i\gamma\omega} \tag{338}$$

which shows that there will be a shift in the transition's frequency to $\sqrt{\omega_0^2 + \omega_p^2 f_{12}}$ (more advanced treatments of this problem don't always include the oscillator strength in this shift).

Personally, I find the proof above very confusing and so I give a more general proof for the transmission of a uniaxial etalon in sec.23. This finds that

$$T_p = 1 + \Im\left[\frac{\varepsilon_b}{\varepsilon_{zz}}\right] n_0 K \frac{\sin^2\theta}{\cos\theta} d \tag{339}$$

which gives the same result.

There is a further complication however, there is a non-apparent subtly in how the ISBT susceptibility is defined. Total susceptibility should be unitless but for the equations to achieve this, we must use a 3-dimensional charge density for the quantum well. Therefore we often define the charge density as

$$N = \frac{N_{2D}}{L_{QW}}$$

We note that the L_{QW} will cancel out on the top of the eqn.329 when we write out the equations in full but not on the bottom which decides the depolarisation shift. Moreover, the wavefunctions in the quantum well are not tophat functions filling the quantum-well layer. This means that we shouldn't really use L_{QW} for the ISBT but we should find an effective length L_{ij}^{eff} which is somehow defined by the initial and final wavefunctions $\psi_i(z)$ and $\psi(z)$ of the ISBT (see the notes for my pyQW project and also [34, 33, 36]).

$$N = \frac{N_{2D}}{L_{ij}^{eff}} \tag{340}$$

This length is less than L_{QW}; for my calculations, I have often found $L_{ij}^{eff} \sim 0.6 L_{QW}$ (see sec.26).

Finally, it is important to be aware of these complications but a transfer matrix model of our quantum well system should automatically include all of these effects as long as we define our dielectric constants carefully. Then all that we need to do is put the dielectric constant of the quantum well as $\varepsilon_{zz} = \varepsilon_{well} + \chi_{ISBT}$. However, this is covered in more detail in sec.26.

23 An Etalon of this particular case of uniaxial material

For studying intersubband absorption in quantum wells, we have a lossy uniaxial material with optical axis along the growth direction. Therefore we are interested in studying (and calculating) the transmission of an extra-ordinary wave through a uniaxial layer.

We can start from the equation for a etalon of our uniaxial layer type given in sec.15. This equation can also easily be rederived using the transfer matrix formulism given in sec.17

$$t = \frac{t_{01} t_{12} e^{i\delta_1}}{1 + r_{01} r_{12} e^{2i\delta_1}} \tag{341}$$

Our aim is to simplify this formula as much as possible.

For this derivation, we will use the following equations for the Fresnel coefficients for a TM (p) polarised wave:

$$t_{01} = \frac{2\varepsilon_{xx} k_z^{(0)}}{\varepsilon_{xx} k_z^{(0)} + \varepsilon_0' k_z^{(1)}} \frac{n_0}{n_1} \tag{342}$$

$$r_{01} = \frac{\varepsilon_{xx} k_z^{(0)} - \varepsilon_0' k_z^{(1)}}{\varepsilon_{xx} k_z^{(0)} + \varepsilon_0' k_z^{(1)}} \tag{343}$$

$$t_{12} = \frac{2\varepsilon_2 k_z^{(1)}}{\varepsilon_2 k_z^{(1)} + \varepsilon_{xx} k_z^{(2)}} \frac{n_1}{n_2} \tag{344}$$

$$r_{12} = \frac{\varepsilon_2 k_z^{(1)} - \varepsilon_{xx} k_z^{(2)}}{\varepsilon_2 k_z^{(1)} + \varepsilon_{xx} k_z^{(2)}} \tag{345}$$

$$\delta_1 = k_z^{(1)} d_1 \tag{346}$$

where d_1 is the thickness of the layer and ε_{xx} is the dielectric constant of the layer for electric fields parralel to the interface (ε_0' is the dielectric constant of the zeroth layer not the vacuum constant - sorry about the bad notation). The k-vector components are calculated from eqn.298.

$$k_z^2 = \varepsilon_{xx} K^2 - \frac{\varepsilon_{xx}}{\varepsilon_{zz}} k_x^2 \tag{347}$$

where $K = \frac{\omega}{c}$ and k_x is constant across the layers.

$$k_x = k_0 \sin\theta_0 = \frac{n_0 \omega}{c} \sin\theta \tag{348}$$

$$k_z^{(0)} = \frac{n_0 \omega}{c} \cos\theta \tag{349}$$

where θ is the angle of incidence. Finally, in this case, we have

$$T = \left(\frac{k_z^{(2)}}{k_z^{(0)}} \right) |t|^2 \tag{350}$$

Firstly, if $\varepsilon_0' = \varepsilon_2 = \varepsilon_b$, we can simplify the transmission of the uniaxial slab down to

$$t_p = \left(\cos\delta_1 - i \sin\delta_1 \left(\frac{\varepsilon_{xx}^2 k_z^{(0)2} + \varepsilon_b^2 k_z^{(1)2}}{2\varepsilon_{xx}\varepsilon_b k_z^{(0)} k_z^{(1)}} \right) \right)^{-1} \tag{351}$$

we can write

$$\frac{\varepsilon_{xx}^2 k_z^{(0)2} + \varepsilon_b^2 k_z^{(1)2}}{2\varepsilon_{xx}\varepsilon_b k_z^{(0)} k_z^{(1)}} = \frac{1}{2} \left(\frac{\varepsilon_{xx} k_z^{(0)}}{\varepsilon_b k_z^{(1)}} + \frac{\varepsilon_b k_z^{(1)}}{\varepsilon_{xx} k_z^{(0)}} \right) \tag{352}$$

and also

$$\begin{aligned}\left(\frac{k_z^{(1)}}{k_z^{(0)}}\right)^2 &= \frac{\varepsilon_{xx}K^2 - \frac{\varepsilon_{xx}}{\varepsilon_{zz}}k_x^2}{\varepsilon_b K^2 - k_x^2} \\ &= \frac{\varepsilon_{xx} - \frac{\varepsilon_{xx}}{\varepsilon_{zz}}\varepsilon_b \sin^2\theta}{\varepsilon_b - \varepsilon_b \sin^2\theta} \\ &= \frac{\varepsilon_{xx}}{\varepsilon_b}\frac{1 - \frac{\varepsilon_b}{\varepsilon_{zz}}\sin^2\theta}{\cos^2\theta} \end{aligned} \tag{353}$$

So if we further assume that $\varepsilon_{xx} = \varepsilon_b$, we write

$$\frac{\varepsilon_{xx}^2 k_z^{(0)2} + \varepsilon_b^2 k_z^{(1)2}}{2\varepsilon_{xx}\varepsilon_b k_z^{(0)} k_z^{(1)}} \Rightarrow \frac{1}{2}\left(\frac{1 - \frac{\varepsilon_b}{\varepsilon_{zz}}\sin^2\theta + \cos^2\theta}{\sqrt{1 - \frac{\varepsilon_b}{\varepsilon_{zz}}\sin^2\theta}\cos\theta}\right) \tag{354}$$

Now, we can often assume that $\delta_1 \ll 1$, we can also write

$$\delta_1 = \sqrt{\varepsilon_{xx}}K\left(1 - \frac{\varepsilon_b}{\varepsilon_{zz}}\sin^2\theta\right)^{1/2} d \tag{355}$$

so that

$$t_p = \left(1 - in_0K\frac{d}{2}\left(\frac{1 - \frac{\varepsilon_b}{\varepsilon_{zz}}\sin^2\theta + \cos^2\theta}{\cos\theta}\right)\right)^{-1} \tag{356}$$

(where we have continued to assume that $\sqrt{\varepsilon_{xx}} = n_0$).

Since ε_{zz} is imaginary that complicates things slightly. The transmission coefficient for the slab is

$$T_p = \left(\left|1 - in_0K\frac{d}{2}\left(\frac{1 - \frac{\varepsilon_b}{\varepsilon_{zz}}\sin^2\theta + \cos^2\theta}{\cos\theta}\right)\right|^2\right)^{-1} \tag{357}$$

Putting

$$\frac{\varepsilon_b}{\varepsilon_{zz}} = z + iz' \tag{358}$$

gets

$$T_p = \left(\left|1 - n_0K\frac{d}{2}\left(\frac{z'\sin^2\theta}{\cos\theta}\right) - in_0K\frac{d}{2}\left(\frac{1 - z\sin^2\theta + \cos^2\theta}{\cos\theta}\right)\right|^2\right)^{-1} \tag{359}$$

$$T_p = \left(\left\{1 - n_0K\frac{d}{2}\left(\frac{z'\sin^2\theta}{\cos\theta}\right)\right\}^2 + \left\{n_0K\frac{d}{2}\left(\frac{1 - z\sin^2\theta + \cos^2\theta}{\cos\theta}\right)\right\}^2\right)^{-1} \tag{360}$$

If we assume that $z = 1$, the second term becomes $\{n_0Kd\cos\theta\}^2$ and as before, we can assume that this is small. So that now we have

$$T_p = \left(1 - n_0K\frac{d}{2}\left(\frac{z'\sin^2\theta}{\cos\theta}\right)\right)^{-2} \tag{361}$$

expand this using

$$\frac{1}{(1+x)^2} \approx 1 - 2x + 3x^2 - 4x^3 \ldots$$

$$T_p = 1 + 2n_0K\frac{d}{2}\left(\frac{z'\sin^2\theta}{\cos\theta}\right)\ldots \tag{362}$$

noting that I assume that z' will turn out to be negative in order that the transmission is less than unity. So finally, we have

$$T_p = 1 + \Im\left[\frac{\varepsilon_b}{\varepsilon_{zz}}\right] n_0K\frac{\sin^2\theta}{\cos\theta}d \tag{363}$$

24 Absorbing Uniaxial Layer

Consider a Lorentz oscillator model of a dielectric.

$$\varepsilon = \varepsilon_b \left(1 + \frac{\omega_p^2 f}{\omega_0^2 - \omega^2 - 2i\gamma\omega} \right) \tag{364}$$

which can be written more simply as

$$\varepsilon = \varepsilon_b \left(1 + \frac{\omega_p^2 f}{2\omega_0'} \frac{1}{\omega_0' - \omega - i\gamma} \right) \tag{365}$$

using the rotating wave approximation, where $\omega_0'^2 = \omega_0^2 - \gamma^2$. We can approximate the absorption of such a medium using

$$\alpha = \frac{2\kappa\omega}{c} = \frac{\varepsilon'\omega}{cn_b} \tag{366}$$

where we have neglected the contribution of transitions contribution to the refractive index. ε' is the imaginary part of the dielectric. Therefore we have

$$\alpha = \frac{n_b \omega \omega_p^2 f}{2\omega_0' c} \frac{\gamma}{(\omega_0' - \omega)^2 + \gamma^2} \tag{367}$$

Now if we have an anisotropic slab with a transition given by

$$\varepsilon_{zz} = \varepsilon_b \left(1 + \frac{\omega_p^2 f}{\omega_0^2 - \omega^2 - 2i\gamma\omega} \right) \tag{368}$$

Then we calculate the absorption using

$$T_p = 1 + \Im \left[\frac{\varepsilon_b}{\varepsilon_{zz}} \right] n_b \frac{\omega}{c} \frac{\sin^2\theta}{\cos\theta} d \tag{369}$$

or

$$\alpha d = -\Im \left[\frac{\varepsilon_b}{\varepsilon_{zz}} \right] n_b \frac{\omega}{c} \frac{\sin^2\theta}{\cos\theta} d \tag{370}$$

$$\frac{\varepsilon_b}{\varepsilon_{zz}} = \left(1 + \frac{\omega_p^2 f}{\omega_0^2 - \omega^2 - 2i\gamma\omega} \right)^{-1} \tag{371}$$

$$\frac{\varepsilon_b}{\varepsilon_{zz}} = 1 - \frac{\omega_p^2 f}{\omega_0^2 + \omega_p^2 f - \omega^2 - 2i\gamma\omega} \tag{372}$$

which we can simplify like before to give

$$\frac{\varepsilon_b}{\varepsilon_{zz}} = \left(1 - \frac{\omega_p^2 f}{2\omega_0''} \frac{1}{\omega_0'' - \omega - i\gamma} \right) \tag{373}$$

where $\omega_0''^2 = \omega_0^2 - \gamma^2 + \omega_p^2 f$. So we get

$$\Im \left[\frac{\varepsilon_b}{\varepsilon_{zz}} \right] = -\frac{\omega_p^2 f}{2\omega_0''} \frac{\gamma}{(\omega_0'' - \omega)^2 + \gamma^2} \tag{374}$$

which gives us

$$\alpha = \frac{n_b \omega \omega_p^2 f}{2\omega_0'' c} \frac{\sin^2\theta}{\cos\theta} \frac{\gamma}{(\omega_0'' - \omega)^2 + \gamma^2} \tag{375}$$

which is similar to the simpler absorption coefficient given earlier (eqn.329) except that the transition frequency has been increased. This is the depolarisation shift [36].

This equation is normally derived using qualitative arguements [36], so the $\sin^2\theta$ factor is attributed to projection of the light wave's electric field vector onto the vertical direction (which is the vector direction of the anisotropic transition). Then the $1/\cos\theta$ is attributed to the effective change in layer thickness with angle of incidence. The depolarisation shift is then added using an arguement based upon the effective local electric field seen by the transition, hence the depolarisation shift can be called a collective excitation effect since it is due to the mutual coulombic interactions of the electrons of the transitions.

The dependence upon the reciprocal of the dielectric constant has interesting similarities with the calculation of the effective dielectric constant for a layered medium (see 6.6 and next section).

25 Effective Medium Approach

In the case of ISBT, the transitions occur inside a medium that already has a dielectric constant. Moreover, the quantum wells are much smaller than the wavelength of light and a sample will normally include tens of QWs; in this situation we can usefully consider all the quantum wells and their barriers as one effective medium with an effective uniaxial dielectric constant. To find the dielectric constant we can use the equations from sec.6.6.

For the electric field parallel to the interfaces, we get

$$\varepsilon_{xx} = \sum_i f_i \varepsilon_i$$

where f_i is the fractional width of each layer and ε_i is its dielectric constant.

Whereas for electric field perpendicular to the interfaces (along the growth direction), the dielectric constant is given by

$$\frac{1}{\varepsilon_{zz}} = \sum_i \frac{f_i}{\varepsilon_i}$$

These become

$$\varepsilon_{xx} = \frac{1}{L_{period}} \left(L_{QW}\varepsilon_{GaAs} + L'_{eff}\varepsilon_{GaAs}\left(\varepsilon_{Drude} - 1\right) + \left(L_{period} - L_{QW}\right)\varepsilon_{AlGaAs} \right)$$

and

$$\frac{1}{\varepsilon_{zz}} = \frac{1}{L_{period}} \left(\left(L_{QW} - L_{eff}\right)\frac{1}{\varepsilon_{GaAs}} + L_{eff}\frac{1}{\varepsilon_{GaAs}\varepsilon_{ISBT}} + \left(L_{period} - L_{QW}\right)\frac{1}{\varepsilon_{AlGaAs}} \right)$$

where L_{QW} is the thickness of the QW layer, L_{period} is the period thickness (QW+barrier) and L_{eff} is the effective thickness of the intersubband transition (this topic is covered in more depth in sec.26 but for now consider that it is approximately $0.6L_{QW}$), it is likely that $L'_{eff} = L_{eff}$. In this formulism, ε_{ISBT} doesn't include the depolarisation shift as it comes out naturally in this derivation and it doesn't include the factor of ε_{GaAs}.

Often we can say that $\varepsilon_{GaAs} = \varepsilon_{AlGaAs}$ so that we have

$$\varepsilon_{xx} = \varepsilon_{GaAs}\left(1 + \frac{L'_{eff}}{L_{period}}\left(\varepsilon_{Drude} - 1\right)\right)$$

$$\frac{1}{\varepsilon_{zz}} = \frac{1}{\varepsilon_{GaAs}}\left(1 - \frac{L_{eff}}{L_{period}}\left(\frac{\varepsilon_{ISBT} - 1}{\varepsilon_{ISBT}}\right)\right)$$

and so

$$\varepsilon_{zz} = \varepsilon_{GaAs}\varepsilon_{ISBT}\left(\varepsilon_{ISBT} - \frac{L_{eff}}{L_{period}}\left(\varepsilon_{ISBT} - 1\right)\right)^{-1}$$

26 Quantum Well Intersubband Tranitions

More advanced treatments of quantum well transitions calculate the depolarisation shift quantum mechanically. This is more accurate in this case because the QW wavefunctions are not sharply deliminated by the bandstructure interfaces. This results in changes to the values of $\omega_p^2 f$ or can equivalently be considered to be calculating the effective thickness of the transition layer. Since the calculation has already taken account of the depolarisation shift, we need to incorporate the results so that the depolarisation shift is not applied for a second time. Many of the models calculate the conductivity tensor rather than a dielectric constant [13, 34, 33].

We have from [13], an effective medium for the multiple quantum well layers with an effective dielectric tensor. The final equations to apply are

$$\frac{1}{\varepsilon_{zz}} = \frac{(1-ff)}{\varepsilon_b} + \frac{ff}{\varepsilon_w} - \frac{i\tilde{\sigma}_{zz}^{(2D)(MQW)}(\omega)}{\varepsilon_0 \varepsilon_w^2 \omega L_{MQW}} \tag{376}$$

$$\varepsilon_{xx} = (1-ff)\,\varepsilon_b + ff\varepsilon_w + \frac{i\sigma_{xx}^{(2D)(MQW)}(\omega)}{\varepsilon_0 \omega L_{MQW}} \tag{377}$$

(where I have tried to change from the cgs. units used in the paper to SI units). ff is the filling factor of the well layers within the MQW slab, ε_b (ε_w) are dielectric constants of the barriers (wells) and $\tilde{\sigma}_{zz}^{(2D)(MQW)}(\omega)$ is a component of the conductivity tensor that already includes the depolarisation effect, actually it is the integrated conductivity over the width of the MQW slab. If we consider each QW to be separated from it's neighbours (i.e. if we don't have a superlattice), then $\tilde{\sigma}_{zz}^{(2D)(MQW)}(\omega) = K\tilde{\sigma}_{zz}^{(2D)}(\omega)$, where K is the number of QWs in the slab and $\tilde{\sigma}_{zz}^{(2D)}(\omega)$ is the conductivity for a single quantum well. If L_{MQW} is the thickness of the entire slab and L_{SQW} is the thickness of one period, we can also write

$$\begin{aligned}
\frac{1}{\varepsilon_{zz}} &= \frac{1}{L_{SQW}}\left\{\frac{(L_{SQW}-L_{QW})}{\varepsilon_b} + \frac{L_{QW}}{\varepsilon_w} - \frac{i\tilde{\sigma}_{zz}^{(2D)}(\omega)}{\varepsilon_0 \varepsilon_w^2 \omega}\right\} \\
\varepsilon_{xx} &= \frac{1}{L_{SQW}}\left\{(L_{SQW}-L_{QW})\,\varepsilon_b + L_{QW}\varepsilon_w + \frac{i\sigma_{xx}^{(2D)}(\omega)}{\varepsilon_0 \omega}\right\}
\end{aligned}$$

If $\varepsilon_b = \varepsilon_w$, this simplifies to

$$\begin{aligned}
\frac{1}{\varepsilon_{zz}} &= \frac{1}{\varepsilon_b}\left\{1 - \frac{i\tilde{\sigma}_{zz}^{(2D)}(\omega)}{L_{SQW}\varepsilon_0 \varepsilon_b \omega}\right\} \\
\varepsilon_{xx} &= \varepsilon_b\left\{1 + \frac{i\sigma_{xx}^{(2D)}(\omega)}{L_{SQW}\varepsilon_0 \varepsilon_b \omega}\right\}
\end{aligned}$$

Ando[34, 33] has shown that $\tilde{\sigma}_{zz}^{(2D)}(\omega)$ has the form

$$\tilde{\sigma}_{zz}^{(2D)}(\omega) = (-i\omega)\frac{e^2}{m^*}\sum_{\beta}^{\Lambda}\left\{\frac{[\tilde{N}\tilde{f}]_\beta}{\tilde{\omega}_\beta^2 - \omega^2 - 2i\gamma\omega}\right\}$$

where $\tilde{\omega}_\beta$ are the transitions frequency (after depolarisation shift) and $\left[\tilde{N}\tilde{f}\right]_\beta$ are their effective values of the product of the oscillator strength and population difference. Strictly speaking, if there are multiple intersubband transitions, they will interact through their depolarisation fields and this will shift their transitions frequencies and oscillator strengths [34, 37]. For a single transition, we have

$$\begin{aligned}
\tilde{\omega}_{01}^2 &= \omega_{01}^2 + \omega_{p01}^2 \\
\left[\tilde{N}\tilde{f}\right]_{01} &= \frac{2m^*\omega_{01}z_{01}^2}{\hbar}\Delta N_{01}
\end{aligned}$$

where

$$\omega_{p_{ij}}^2 = \frac{\Delta N_{ij} e^2}{m^* \varepsilon_r \varepsilon_0 L_{ij}^{eff}}$$

$$L_{ij}^{eff} = \frac{\hbar}{2S_{ijij} m^* \omega_{ij}}$$

$$S_{jimn} = -\frac{1}{\hbar\omega_{ij}}\frac{1}{\hbar\omega_{mn}}\left(-\frac{\hbar^2}{2m^*}\right)^2\int_0^\infty \left(\frac{d\xi_i}{dz}\xi_j - \xi_i\frac{d\xi_j}{dz}\right)\left(\frac{d\xi_m}{dz}\xi_n - \xi_m\frac{d\xi_n}{dz}\right) dz \tag{378}$$

where ξ are the level wavefunctions.

We can in general calculate the absorption of the system using

$$\Re\left[\tilde{\sigma}_{zz}^{(2D)}(\omega)\right] \approx \omega\frac{e^2}{m^*}\sum_\beta^\Lambda\left\{\frac{[\tilde{N}\tilde{f}]_\beta}{2\tilde{\omega}'_\beta}\frac{\gamma}{\left(\tilde{\omega}'_\beta-\omega\right)^2+\gamma^2}\right\}$$

where $\tilde{\omega}'^2_\beta = \tilde{\omega}^2_\beta - \gamma^2$. Therefore the absorption in this case becomes

$$\begin{aligned}
\alpha &= \frac{\Re\left[\tilde{\sigma}_{zz}^{(2D)}(\omega)\right]}{L_{SQW}\varepsilon_0 n_b c}\frac{\sin^2\theta}{\cos\theta} \\
\alpha &= \frac{1}{L_{SQW}\varepsilon_0\varepsilon_b}n_b\frac{\omega}{c}\frac{\sin^2\theta}{\cos\theta}\frac{e^2}{m^*}\sum_\beta^\Lambda\left\{\frac{[\tilde{N}\tilde{f}]_\beta}{2\tilde{\omega}'_\beta}\frac{\gamma}{\left(\tilde{\omega}'_\beta-\omega\right)^2+\gamma^2}\right\}
\end{aligned}$$

This can be written as

$$\alpha = n_b\frac{\omega}{c}\frac{\sin^2\theta}{\cos\theta}\sum_\beta^\Lambda\left\{\frac{R_\beta^2}{2\omega'_\beta}\frac{\gamma}{\left(\tilde{\omega}'_\beta-\omega\right)^2+\gamma^2}\right\}$$

where

$$R_\beta^2 = \frac{e^2}{m^*\varepsilon_0\varepsilon_b L_{SQW}}[\tilde{N}\tilde{f}]_\beta$$

The total absorption of the layer will then be

$$\alpha L_{SQW}$$

where L_{SQW} is the quantum well period (well + barrier).

Part V

Strong Light-Matter Coupling: Polaritons

27 A brief introduction

When we consider interactions of light with transitions, phonon or excitons, we may be used to thinking about these effects in terms of absorptions and emmissions. If we want to calculate the absorption/transmissivity of a slab, we might think in terms off Beer's Law

$$I = I_0 e^{-\alpha z} \tag{379}$$

where α is the absorption coefficient, z is the distance travelled and I_0 is the initial intensity. Normally, a transition has an absorption which is wavelength dependent, perhaps it is a peaked function that can be described by a scaled Lorentzian lineshape

$$\alpha(\omega) = \frac{A\gamma^2}{(\omega-\omega_0)^2+\gamma^2} \tag{380}$$

where γ is the full width half maximum of the lineshape and A is the amplitude of the lineshape.

However, when the transition is strongly coupled to the light, we see a new behaviour, this is illustrated in fig.4. The strong coupling between light and another oscillator leads to new 'light-matter modes' that are named polaritons (or 'dressed states'). These mixed modes are generally at frequencies either side of the original frequency of the transition and between these modes there often exists a forbidden zone where

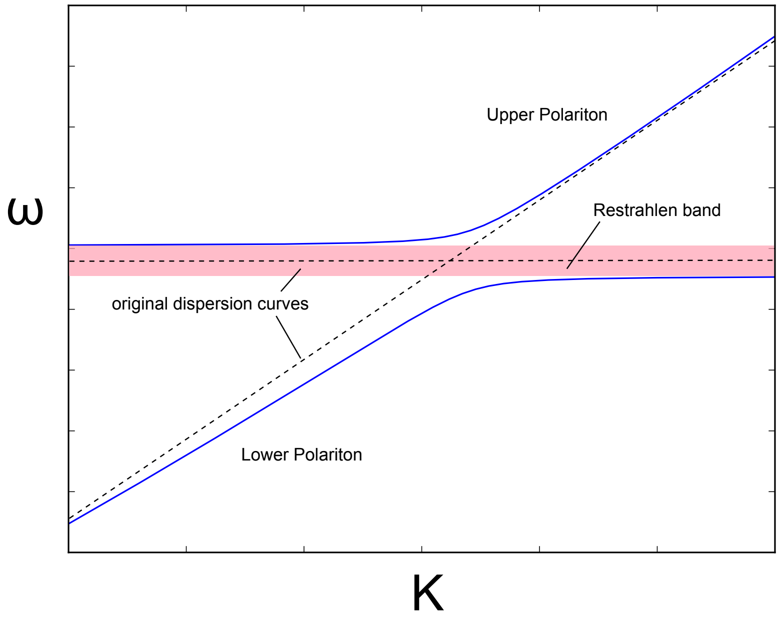


Figure 4: An idealised polariton curve, showing upper and lower bands and the forbidden region. The original dispersion curves are the light line and horizontal line is interacting oscillator or transition. We can see that there is an anti-crossing between the two oscillators.

no waves can propagate; waves will be critically damped in this region. The signature of a polaritonic phenomena is when the light and transition dispersion curves 'anti-cross', meaning that the oscillator and the light line dispersions don't cross as expected but instead will transmute into each other. Although surprising, anti-crossing is a basic phenomena of all coupled oscillators, examples of which can be found in the *normal modes* of coupled-pendulums or in the bandgaps of solid state electronic bandstructure.

Clearly, the transition is no longer operating as an absorption but as an oscillator. In fact, all transitions are fundamentally oscillators but we get drawn into considering them as absorbers under normal conditions due to the rapidity which electrons incoherently scatter/relax down to the lower state again. Equivalently, the quantum mechanics of optical transitions shows us that absorption and stimulated emission are two sides of the same phenomena but that this symmetry is broken by the non-radiative decay (e.g. by phonon emission) or spontanous emission (of a photon) of the electrons in the upper state. However, when the coupling is strong between the transtion and the light, these decoherent mechanisms become progressivly less important and we return to the more fundamental picture of transitions as oscillators. Therefore polaritons are the name given to the normal modes when light is strongly coupled to another oscillator such as a transition or a phonon.

On a more practical note, how do we changes our models to get from an absorption to a description of polaritons? Notice that generally the refractive index is assumed to be constant throughout a transition/absorption but this is an approximation, the refractive index must also be affected by the transition[9]; for the peaked absorption in equ.380, the refractive index is

$$n\left(\omega\right)=\sqrt{1+\frac{1}{2\omega}\frac{B\omega_{p}^{2}\left(\omega_{0}-\omega\right)}{\left(\omega-\omega_{0}\right)^{2}+\gamma^{2}}} \tag{381}$$

where B is a constant related to A. The incorporation of the refractive index into our modelling of the interaction of a transition with light naturally leads to the possibility of polaritonic effects.

We can also model a polariton by beginning with a dielectric constant derived from a Lorentz Oscillator model

$$\varepsilon_{r}=1+\frac{\omega_{p}^{2}}{\omega_{0}^{2}-\omega^{2}-2i\gamma\omega} \tag{382}$$

[9]It can be shown that changes in absorption must be accompanied by changes in refractive index via the Kramers Kronig relations (see 2.2).

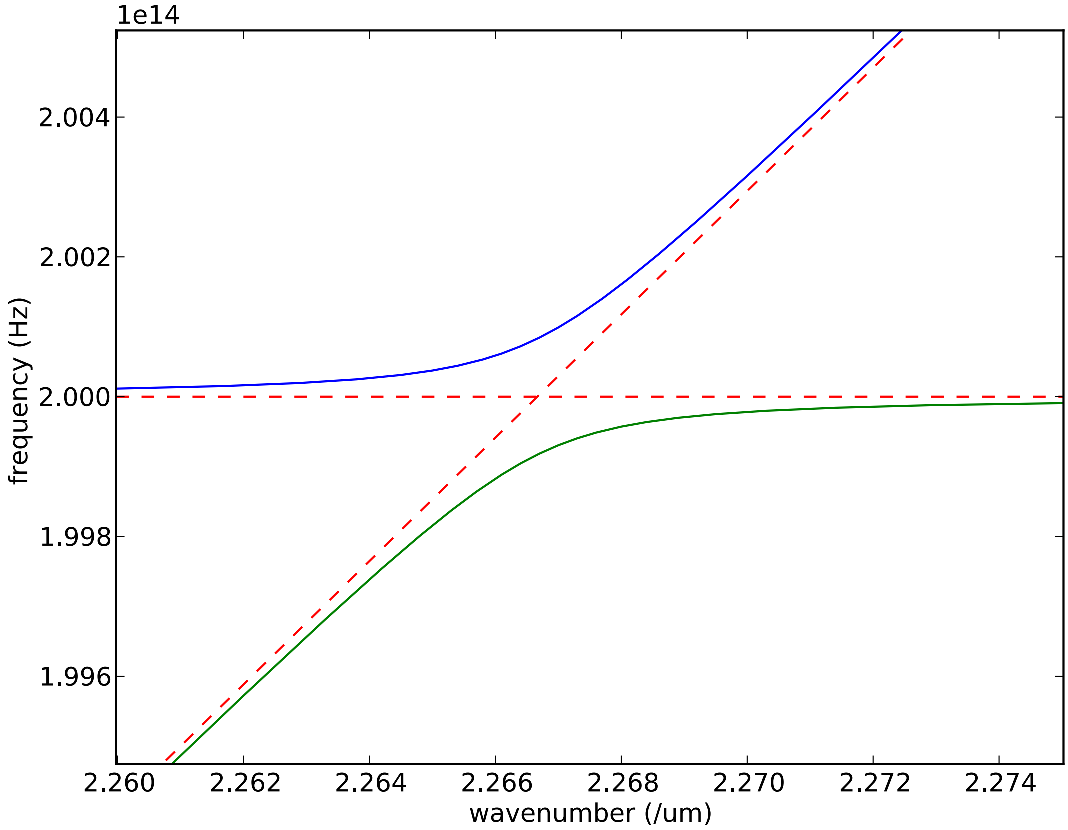


Figure 5: A plot of eqn.388 with $\varepsilon_r(\infty) = 3.4^2$, $\omega_0 = 2 \times 10^{14}$ Hz (9.4 um)and $\omega_p = 5.64 \times 10^{11}$ Hz .

We can see that this explicitly considers the transition as an oscillator. For simplicity we will set $\gamma = 0$ but we will include a background contribution to the dielectric constant (i.e. the refractive index of the material); leading to

$$\varepsilon_r = \varepsilon_r(\infty) + \frac{\omega_p^2}{\omega_0^2 - \omega^2} \tag{383}$$

Then the electromagnetic wave equation is taken

$$\mathbf{k}^2 = \frac{\varepsilon_r}{c^2}\omega^2 \tag{384}$$

Combining the two equations we get

$$\mathbf{k}^2 = \frac{1}{c^2}\left(\varepsilon_r(\infty) + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\right)\omega^2 \tag{385}$$

which leads to

$$c^2\mathbf{k}^2\left(\omega_0^2 - \omega^2\right) = \left(\varepsilon_r(\infty)\,\omega_0^2 - \varepsilon_r(\infty)\,\omega^2 + \omega_p^2\right)\omega^2 \tag{386}$$

$$\varepsilon_r(\infty)\,\omega^4 - \left(c^2\mathbf{k}^2 + \varepsilon_r(\infty)\,\omega_0^2 + \omega_p^2\right)\omega^2 + c^2\mathbf{k}^2\omega_0^2 = 0 \tag{387}$$

giving

$$\omega^2 = \frac{\left(c^2\mathbf{k}^2 + \varepsilon_r(\infty)\,\omega_0^2 + \omega_p^2\right) \pm \sqrt{\left(c^2\mathbf{k}^2 + \varepsilon_r(\infty)\,\omega_0^2 + \omega_p^2\right)^2 - 4\varepsilon_r(\infty)\,c^2\mathbf{k}^2\omega_0^2}}{2\varepsilon_r(\infty)} \tag{388}$$

Fig.5 plots out this equation. There is an energy gap where no modes exist which can be observed in a sample as a band of high reflectivity, for interactions involving optical phonons, this is known as a Restrahlen band.

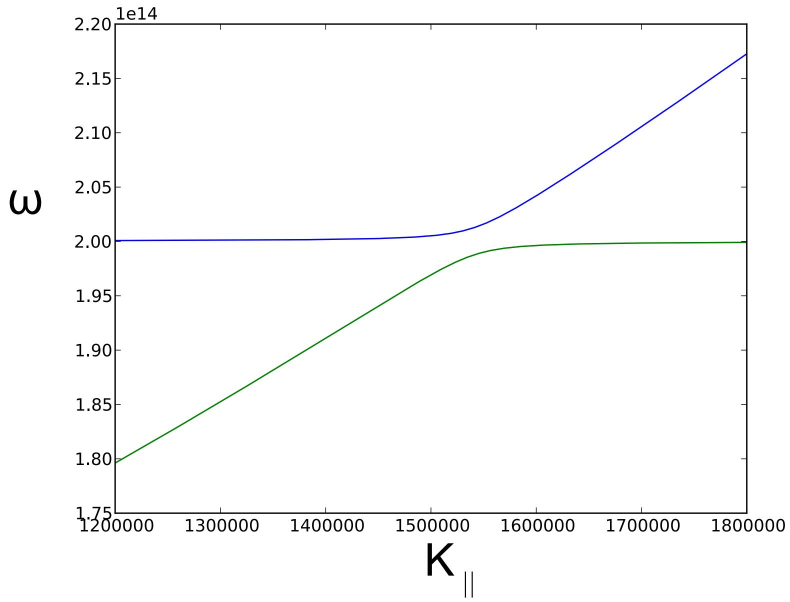


Figure 6: This plot was created using equation 394 with $\omega_{ISBT} = 2 \times 10^{14}$, $d = 4 \times 10^{-6}$, $n_{cavity} = 3.3$, $p = 1$ and $A = 5 \times 10^{26}$.

28 Coupled Oscillators

Considering a problem with two coupled oscillators is illustrative of normal modes. A simple example is

$$\ddot{x} = -\omega_x^2 x + Ay \tag{389}$$
$$\ddot{y} = -\omega_y^2 y + Ax \tag{390}$$

Of course, there could be many variants on this theme with different coupling or damped oscillations. Importantly though, each equation only considers how the oscillator would naturally move and how it is affected by the other oscillator. There isn't a term that explicitly describes how the oscillator's effect on the other oscillator, would affect the first's motion. That's (maybe) because this is a force picture rather than an energy picture. Later in the quantum mechanical descriptions, one will explicitly consider the energy injected and extracted by the other oscillator but we will be thinking in terms of energy rather than force!

To solve these coupled differential equations, we look for normal modes, where both $x$and y are proportional to $e^{-i\omega t}$ and then solve the resulting simultaneous equation. ie

$$\left(\omega_x^2 - \omega^2\right) x_0 = Ay_0 \tag{391}$$
$$\left(\omega_y^2 - \omega^2\right) y_0 = Ax_0 \tag{392}$$

So

$$\left(\omega_y^2 - \omega^2\right)\left(\omega_x^2 - \omega^2\right) = A^2 \tag{393}$$

which has solutions

$$\omega^2 = \frac{\omega_y^2 + \omega_x^2 \pm \sqrt{\left(\omega_y^2 - \omega_x^2\right)^2 + A^2}}{2} \tag{394}$$

In fact, if we put $\omega_x = \omega_{ISBT}$ and $\omega_y = \frac{c}{n_{cavity}}\sqrt{(\frac{2\pi p}{d})^2 + k_{\|}^2}$ which is the dispersion of a slab waveguide (optical cavity), we can create a plot very similar to the classic polariton anticrossing diagram as shown in fig. 6.

29 Microcavity Polaritons

slab cavity - formula for w wrt. angle of incidence

30 Zero-Dimensional Microcavity Polaritons

Consider that we have a microcavity cavity that is confined in all 3 dimensions[35] so that its optical modes are quantised. Its modes still satisfy a wave equation

$$c^2k_c^2 = \varepsilon_b\omega_c^2$$

where ε_b is the dielectric constant of the medium within the cavity (and can be frequency dependent), but the boundary conditions of the cavity will quantise the possible values of k_c and ω_c.

If we now place our transition inside the cavity, we will have a new wave equation

$$c^2k^2 = \varepsilon_r\omega^2$$

Now, if we assume that the cavity resonances are not altered too much by the transition ... we can say that we know the allowed values of k and so can write

$$\varepsilon_b\omega_c^2 = \varepsilon_r\omega^2$$

the solutions of this equation will be the new modes of the system. For a Lorentz oscillator, we will have

$$\varepsilon_b\omega_c^2 = \varepsilon_b\left(1 + \frac{\omega_p^2 f_0}{\omega_0^2 - \omega^2 - 2i\gamma\omega}\right)\omega^2$$

Dropping the broadening term, we get the following dispersion equation

$$\left(\omega_c^2 - \omega^2\right)\left(\omega_0^2 - \omega^2\right) = \omega_p^2 f_0\omega^2$$

For the multiple quantum well structures, the dielectric is actually different from a Lorentz oscillator. We have

$$\frac{1}{\varepsilon_{zz}} = \frac{1}{\varepsilon_b}\left\{1 - \sum_{\gamma}^{\Lambda}\left\{\frac{R_\gamma^2}{\omega_\gamma^2 - \omega^2}\right\}\right\}$$

We will also calculate the effective dielectric constant for a layered medium which is not completely filled with quantum well structures

$$\frac{1}{\varepsilon_{zz}} = \frac{(1 - f_w)}{\varepsilon_b} + \frac{f_w}{\varepsilon_b}\left\{1 - \sum_{\gamma}^{\Lambda}\left\{\frac{R_\gamma^2}{\omega_\gamma^2 - \omega^2}\right\}\right\}$$

where f_w is the fraction of the cavity filled with quantum wells. Leading to

$$\frac{1}{\varepsilon_{zz}} = \frac{1}{\varepsilon_b}\left\{1 - \sum_{\gamma}^{\Lambda}\left\{\frac{R_\gamma^2 f_w}{\omega_\gamma^2 - \omega^2}\right\}\right\}$$

We have the following polariton dispersion

$$\left(1 - \sum_{\gamma}^{\Lambda}\left\{\frac{R_\gamma^2 f_w}{\omega_\gamma^2 - \omega^2}\right\}\right)\omega_c^2 = \omega^2$$

For a single transition, we have

$$\left(\omega_0^2 - \omega^2\right)\left(\omega_c^2 - \omega^2\right) = R_0^2 f_w\omega_c^2$$

Since $R_0^2 \approx \omega_p^2 f_0$, we have

$$\left(\omega_0^2 - \omega^2\right)\left(\omega_c^2 - \omega^2\right) = f_0 f_w\omega_p^2\omega_c^2$$

which is the equation given in the papers[35].

For many transitions

$$\omega_c^2 - \omega_c^2\sum_{\gamma}^{\Lambda}\left\{\frac{R_\gamma^2 f_w}{\omega_\gamma^2 - \omega^2}\right\} = \omega^2$$

We would need to numerically find the roots of this equation. The should be $\Lambda + 1$ solutions.

A Bibliography

References

[1] J. Peatross and M. Ware. Physics of light and optics, 2011. available at optics.byu.edu.

[2] E. Hecht. *Optics*. Addison-Wesley world student series. Addison-Wesley, 1998.

[3] D.H. Staelin, A.W. Morgenthaler, and J.A. Kong. *Electromagnetic Waves.* An Alan R. Apt book. Prentice Hall PTR, 1994.

[4] Hugh Angus Macleod. *Thin film optical filters*. Taylor & Francis, 2001.

[5] Max Born, Emil Wolf, and AB Bhatia. *Principles of optics: electromagnetic theory of propagation, interference and diffraction of light, 7th (expanded) ed.* Cambridge U. Press, Cambridge, UK, 1999.

[6] Koji Ohta and Hatsuo Ishida. Matrix formalism for calculation of electric field intensity of light in stratified multilayered films. *Applied optics*, 29(13):1952–1959, 1990.

[7] Steven J Byrnes. fresnel manual, March 2012.

[8] A.M. Fox. *Optical Properties of Solids*. Oxford Master Series in Physics Series. Oxford University Press, Incorporated, 2001.

[9] P G Etchegoin, E C Le Ru, and M Meyer. An analytic model for the optical properties of gold. *The journal of chemical physics*, 125:164705, 2006.

[10] N Laman and D Grischkowsky. Terahertz conductivity of thin metal films. *Appl. Phys. Lett.*, 93:051105, 2008.

[11] Aspnes. Local field effects and effective medium theory a microscopic perspective. *Am. J. Phys*, 50(8):705–709, 1981.

[12] R.P. Feynman, R. Leighton, and M. Sands. *The Feynman Lectures on Physics. The Definitive and Extended Edition*. Addison Wesley, 2005.

[13] M. Zaluzny and C Nalewajko. Coupling of infrared radiation to intersubband transitions in multiple quantum wells: The effective-medium approach. *Physical Review B: Condensed Matter and Materials Physics*, 59(20):13 043, 1999.

[14] J Leng, J Opsal, H Chu, M Senko, and D E Aspnes. Analytical representations of the dielectric functions of materials for device and structural modeling. *Thin Solid Films*, 313:132–136, 1998.

[15] G. Brooker. *Modern Classical Optics*. Oxford Master Series in Physics. OUP Oxford, 2003.

[16] E. Rosencher and B. Vinter. *Optoelectronics*. Cambridge University Press, 2002.

[17] Leif AA Pettersson, Lucimara S Roman, and Olle Inganas. Modeling photocurrent action spectra of photovoltaic devices based on organic thin films. *Journal of Applied Physics*, 86(1):487–496, 1999.

[18] M Claudia Troparevsky, Adrian S Sabau, Andrew R Lupini, and Zhenyu Zhang. Transfer-matrix formalism for the calculation of optical response in multilayer systems: from coherent to incoherent interference. *Optics Express*, 18(24):24715–24721, 2010.

[19] MA Dupertuis, B Acklin, and M Proctor. Generalized energy balance and reciprocity relations for thin-film optics. *JOSA A*, 11(3):1167–1174, 1994.

[20] Berreman. Optics in stratified and anisotropic media 4x4 matrix formulation. *Journal of the Optical Society of America*, 62(4):502–510, 1972.

[21] P. Yeh. *Optical waves in layered media*. Wiley series in pure and applied optics. Wiley, 2005.

[22] P. Yeh. Optics of anisotropic layered and media. *Surface Science*, 96:41–53, 1980.

[23] P. Yeh. Electromagnetic propagation in birefringent layered media. *Journal of the Optical Society of America*, 69(5):742, 1979.

[24] Yeh. Extended jones matrix method. *Journal of the Optical Society of America*, 72(4):507–513, 1982.

[25] Teitler. Refraction in stratified anisotropic media. *J. Opt. Soc. Am.*, 60(6):830–834, 1970.

[26] Chung. 4x4 matrix formulisms for optics in stratified anisotropic media. *J. Opt. Soc. Am.*, 7:703–705, 1984.

[27] Wohler. Faster 4x4 matrix method for uniaxial inhomogeneous media. *J. Opt. Soc. Am.*, 5(9):1554–1557, 1988.

[28] Schubert. Polarization-dependent optical parameters of arbitrarily anisotropic homogeneous layers systems. *Physical Review B*, 53(8):4265–4274, 1996.

[29] Chen. 4x4 and 2x2 matrix formulations for the optics in stratified and biaxial media. *Journal of the Optical Society of America*, 14(11):3125–3134, 1997.

[30] Abdulhalim. Exact 2x2 matrix method for the transmission and reflection at the interface between two arbitariliy orientated biaxial crystals. *Journal of Optics A*, 1:655–661, 1999.

[31] Abdulhalim. Analytic propagation matrix method for linear optics of arbitrary biaxial layered media. *J. Opt. A*, 1:646–653, 1999.

[32] Chen. Efficient and accurate numerical analysis of multilayer planar optical waveguides in lossy anisotropic media. *Optics Express*, 7(8):260–272, 2000.

[33] Tsuneya Ando. Electronic properties of two-dimensional systems. *Reviews of Modern Physics*, 54(2), 1982.

[34] Tsuneya Ando. Intersubband optical absorption in space-charge layers on semiconductor surfaces. *Z. Physik B*, 26:263–272, 1977.

[35] Y. Todorov, A.M. Andrews, R. Colombelli, S. De Liberato, C.Ciuti, P. Klang, G. Strasser, and C. Sirtori. Ultrastrong light-matter and coupling regime and with polariton and dots. *Physical Review Letters*, 105:196402, November 2010.

[36] H.C. Liu. *Intersubband Transitions in Quantum Wells I*, volume vol.65 of *Semiconductors and Semimetals*. academic press, 2000.

[37] A. Delteil, A. Vasanelli, Y. Todorov, C. Feuillet Palma, Renaudat St-Jean, G. Beaudoin, I. Sagnes, and C. Sirtori. Charge-induced coherence and between intersubband and plasmons in a quantum and structure. *Physics Review Letters*, 109:246808, 2012.

[38] JB Williams, MS Sherwin, KD Maranowski, and AC Gossard. Dissipation of intersubband plasmons in wide quantum wells. *Physical Review Letters*, 87(3):037401, 2001.